Proof of the Kalman gain matrix: Apriori-Aposteriori form of the discrete Kalman filter

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1 Kalman filter gain matrix

The Kalman filter on apriori-aposteriori form is given by

$$\bar{y}_k = D\bar{x}_k \quad (1)$$
$$\hat{x}_k = \bar{x}_k + K(y_k - \bar{y}_k), \quad (2)$$
$$\bar{x}_{k+1} = A\hat{x}_k + R_{12}\Delta^{-1}(y_k - \bar{y}_), \quad (3)$$

An alternative derivation of the Kalman filter gain matrix in Eq. (2) is as follows

$$\mathrm{E}(\Delta\hat{x}_k\varepsilon_k^T) = \mathrm{E}((\Delta\bar{x}_k - K\varepsilon_k)\varepsilon_k^T) = 0. \quad (4)$$

And from Eq. (4) we have

$$\mathrm{E}(\Delta\hat{x}_k\varepsilon_k^T) = \mathrm{E}((\Delta\bar{x}_k - K\varepsilon_k)\varepsilon_k^T) = \mathrm{E}(\Delta\bar{x}_k \overbrace{(\Delta\bar{x}_k^T D^T + w_k^T)}^{\varepsilon_k^T} - K\mathrm{E}(\varepsilon_k\varepsilon_k^T)$$
$$= \bar{X}D^T - K\mathrm{E}(\varepsilon_k\varepsilon_k^T) = 0, \quad (5)$$

since $\mathrm{E}(\Delta\bar{x}_k w_k^T) = 0$.
We then get from Eq. (5) that the optimal Kalman filter gain matrix in the filter is given by

$$K = \bar{X}D^T(D\bar{X}D^T + W)^{-1}, \quad (6)$$

where we have used that

$$\Delta = \mathrm{E}(\varepsilon_k \varepsilon_k^T) = D\bar{X}D^T + W. \tag{7}$$

From lecture notes Ch. 2.6.3 main_estim_e2.pdf