Solution Final Exam IIA2217 24/5-24

Task 1

a)

- Putting $x_k = y_k - e_k$ in Eq. (1) and $k := k-1$ gives

$$y_k - e_k = a(y_{k-1} - e_{k-1}) + b\,u_{k-1} + k\,e_{k-1}$$

$$\Downarrow$$

$$y_k = a\,y_{k-1} + b\,u_{k-1} + (k-a)\,e_{k-1} + e_k$$

When $K = a$ we may write as a linear regressor model

$$y_k = \begin{bmatrix} y_{k-1} & u_{k-1} \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} + e_k$$

$$\Downarrow$$

$$y_k = \varphi_k^T \theta_0 + e_k$$

where

$$\theta_0 = \begin{bmatrix} a \\ b \end{bmatrix}, \quad \varphi_k = \begin{bmatrix} y_{k-1} \\ u_{k-1} \end{bmatrix}$$

b)

- Prediction error $\varepsilon_k = y_k - \bar{y}_k$

Predicted output $\bar{y}_k = \varphi_k^T \theta$

$$V_N(\theta) = \frac{1}{N} \sum_{k=1}^{N} (y_k - \varphi_k^T \theta)^T \Lambda (y_k - \varphi_k^T \theta)$$

Solve the gradient

$$g(\theta) = \frac{dV_N(\theta)}{d\theta} = 0$$

We have,

$$g(\theta) = \frac{1}{N}\sum_{k=1}^{N} \frac{d\varepsilon_k}{d\theta}\,\frac{d(\overbrace{\varepsilon^T \Lambda \varepsilon}^{V})}{d\varepsilon} = \frac{1}{N}\sum_{k=1}^{N} -\varphi_k \cdot 2\Lambda\varepsilon$$

$$= -\frac{2}{N}\sum_{k=1}^{N} \varphi_k \Lambda (y_k - \varphi_k^T\theta)$$

$$= -\frac{2}{N}\left(\sum_{k=1}^{N} \varphi_k \Lambda y_k - \sum_{k=1}^{N} \varphi_k \Lambda \varphi_k^T \cdot \theta\right) = 0$$

This ~~gives~~

$$\boxed{\hat{\theta}_N = \left(\sum_{k=1}^{N} \varphi_k \Lambda \varphi_k^T\right)^{-1} \sum_{k=1}^{N} \varphi_k \Lambda y_k}$$

Optimal weighting matrix

$$\Lambda = \left(E(e_k e_k^T)\right)^{-1} = \Delta^{-1}$$

where

$\Delta = E(e_k e_k^T)$, The covariance matrix of the noise process

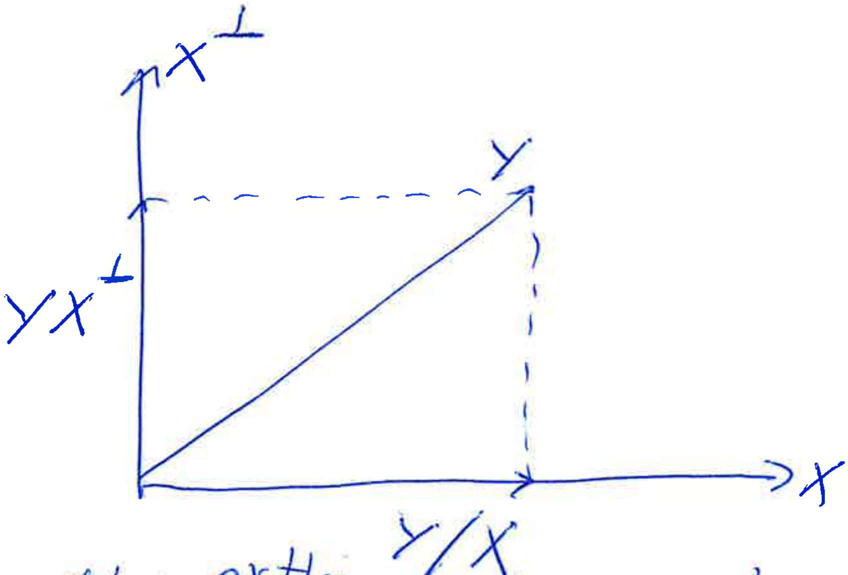
This is the Best Linear Unbiased Estimate (BLUE) estimator.

c)

- Projections Y/X and $YX^\perp$

$$Y/X = YX^T(XX^T)^+X$$

$$YX^\perp = Y - YX^T(XX^T)^+X$$

- 

whee the orthogonal complement $X^\perp$ to X

is $\quad X^\perp = I - X^T(XX^T)^+X$

is such that $\quad XX^\perp = 0$

- $Y = Y/X + YX^\perp$

$$= \cancel{YX^T(XX^T)^+X} + Y \cancel{- YX^T(XX^T)^+X} = Y$$

is correct

d)

- $Y X^T = \Theta X X^T + \cancel{E X^T}$

$$\Downarrow$$

$$\hat{\Theta} = Y X^T (X X^T)^{-1}$$

- Predictor $\bar{Y}$

$$\bar{Y} = \hat{\Theta} X = Y X^T (X X^T)^{-1} X$$

- We have that

$$\bar{Y} = Y X^T (X X^T)^{-1} X = Y / X$$

e) From

$$Y = XB + E$$

we have

$$B_{OLS} = (X^T X)^{-1} X^T Y$$

Task 2

a) $\bar{y}_k = D\bar{x}_k$ Predicted output

$\hat{x}_k = \bar{x}_k + K(y_k - \overbrace{D\bar{x}_k}^{\bar{y}_k})$ Aposteriori estimate

$\bar{x}_{k+1} = A\hat{x}_k + Bu_k$ Update apriori estimate

b) Eliminate $\hat{x}_k$ from above filter

$$\bar{x}_{k+1} = A\overbrace{(\bar{x}_k + K(y_k - D\bar{x}_k))}^{\hat{x}_k} + Bu_k$$

$\Updownarrow$

$$\boxed{\begin{aligned} \bar{x}_{k+1} &= A\bar{x}_k + Bu_k + \tilde{K}\varepsilon_k \\ y_k &= D\bar{x}_k + \varepsilon_k \end{aligned}}$$

Innovations formulation

where $\tilde{K} = AK$

c) • Kalman filter on prediction form

$$\bar{x}_{k+1} = A\bar{x}_k + Bu_k + K(y_k - D\bar{x}_k)$$

$$\bar{y}_k = D\bar{x}_k$$

• Prediction error

$$\varepsilon_k = y_k - \bar{y}_k = y_k - D\bar{x}_k$$

$$A = \begin{bmatrix} 0 & 1 \\ \theta_1 & \theta_2 \end{bmatrix}, \quad B = \begin{bmatrix} \theta_3 \\ \theta_4 \end{bmatrix}, \quad K = \begin{bmatrix} \theta_5 \\ \theta_6 \end{bmatrix}$$

$$D = \begin{bmatrix} 1 & 0 \end{bmatrix} \qquad x_0 = \begin{bmatrix} \theta_7 \\ \theta_8 \end{bmatrix}$$

$$\theta = \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \\ \theta_5 \\ \theta_6 \end{bmatrix}$$ or with x_0, $\theta = [\theta_1\ \theta_2 \ldots \theta_7\ \theta_8]^T$

d)

$$\bar{y}_k = g(\bar{x}_k)$$

$$\hat{x}_k = \bar{x}_k + K(y_k - g(\bar{x}_k))$$

$$\bar{x}_{k+1} = f(\hat{x}_k)$$

Task 3

Task 3

a) $L = J = 2, g = 0, N = 10$ $\quad K = 7$ columns

$$Y_{J+1|L} = \begin{bmatrix} y_3 & y_4 & y_5 & y_6 & y_7 & y_8 & y_9 \\ y_4 & y_5 & y_6 & y_7 & y_8 & y_9 & y_{10} \end{bmatrix}$$

$$U_{J|L+g} = U_{2|2} = \begin{bmatrix} u_2 & & \cdots & & & & u_8 \\ u_3 & u_4 & u_5 & u_6 & u_7 & u_8 & u_9 \end{bmatrix}$$

$$E_{J|L+1} = \begin{bmatrix} e_2 & & \cdots & & & & e_8 \\ e_3 & & \cdots & & & & e_9 \\ e_4 & e_5 & e_6 & e_7 & e_8 & e_9 & e_{10} \end{bmatrix}$$

and so on

b) Remove noise with $\begin{bmatrix} U_{g|L+g} \\ U_{0|J} \\ Y_{0|J} \end{bmatrix} = W$

$$Y_{J|L}/W = \tilde{O}_L X_J/W + \tilde{H}_L^d U_{J|L+g-1}$$

$$Y_{J+1|L}/W = \tilde{A}_L Y_{J|L}/W + \tilde{B}_L U_{J|L+g}$$

because

$U_{J|L+g-1}/W = U_{J|L+g-1}$ and $U_{J|L+g}/W = U_{J|L+g}$

Remove future input terms gives

$$(Y_{J|L}/W) U^{\perp}_{J|L+g} = O_L X_J^g$$

$$(Y_{J+1|L}/W) U^{\perp}_{J|L+g} = \tilde{A}_L (Y_{J|L}/W) U^{\perp}_{J|L+g}$$

Then

$$Z_{J|L} = (Y_{J|L}/W) U^{\perp}_{J|L+g}, \quad Z_{J+1|L} = (Y_{J+1|L}/W) U^{\perp}_{J|L+g}$$

c) Take the SVD of

$$Z_{J|L} = (Y_{J|L}/W)\, U^{\perp}_{J|L+g} = O_L X_J^{d}$$

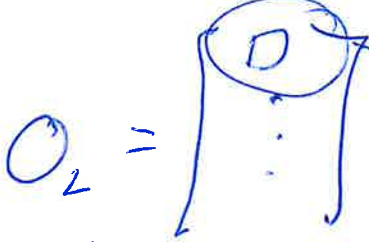
$$USV^T = Z_{J|L} \approx [U_1 \; U_2] \begin{bmatrix} S_1 & 0 \\ 0 & S_2 \end{bmatrix} [V_1 \; V_2]^T$$

$$\approx U_1 S_1 V_1^T$$

where the n large svd's is contained in S_1

- $n = \dim(S_1)$, large singular values
- $O_L = U_1$ output normal realization

$X_J^{d} = S_1 V_1^T$



$\rightarrow D$ matrix i upper part of $O_L = U_1$.

- We have

$$Z_{J+1|L} = \underbrace{O_L A (O_L^T O_L)^{-1} O_L^T}_{\hat{A}_L} \cdot \underbrace{U_1 S_1 V_1^T}_{Z_{J|L}}$$

$$\Downarrow$$

$$Z_{J+1|L} = U_1 A S_1 V_1^T \text{ because } O_L = U_1 \text{ and } U_1^T U = I$$

$$\Downarrow$$

$$\boxed{A = U_1^T Z_{J+1|L} V_1 S_1^{-1}}$$

d) $\mathcal{E}_k = F e_k \Rightarrow e_k = F^{-1}\mathcal{E}_k$

gives $C e_k = C F^{-1} \mathcal{E}_k = K \mathcal{E}_k$

$$\boxed{K = CF^{-1}}$$

e) Signal term

$$\underline{\underline{Y_{j|1} \Big/ \begin{bmatrix} U_{0|j} \\ Y_{0|j} \end{bmatrix} = D X_{j|r} = y^d_{j|1}}}$$

Noise term

$$\underline{\underline{\mathcal{E}_{j|1} = F E_{j|1} = Y_{j|1} - Y_{j|1} \Big/ \begin{bmatrix} U_{0|j} \\ Y_{0|j} \end{bmatrix}}}$$

Hence, we have splitted the outputs in $Y_{j|1}$ into signal and noise

Task 4

Given $H_k = DA^{k-1}B \quad \forall\, k = 1, \dots, 5$

a) $L = 2$

$$H_{2|L} = H_{2|2} = \begin{bmatrix} H_2 & H_3 & H_4 \\ H_3 & H_4 & H_5 \end{bmatrix}$$

$$H_{1|2} = \begin{bmatrix} H_1 & H_2 & H_3 \\ H_2 & H_3 & H_4 \end{bmatrix}$$

b)

$$H_{1|L} = O_L \, C_J$$

$$H_{2|L} = O_L \, A \, C_J$$

C_J is here with $J = 3$

$$C_J = [B \quad AB \quad A^2B]$$

Test

$$O_2 C_3 = \begin{bmatrix} D \\ DA \end{bmatrix} [B \quad AB \quad A^2B] = \left[\begin{array}{c|c|c} DB & DAB & DA^2B \\ \hline DAB & DA^2B & DA^3B \end{array}\right]$$

$$= \begin{bmatrix} H_1 & H_2 & H_3 \\ H_2 & H_3 & H_4 \end{bmatrix}$$

c) Take the SVD of $H_{1|L} = O_L C_J$

$$H_{1|L} = USV^T = [U_1\, U_2]\begin{bmatrix} S_1 & 0 \\ 0 & S_2 \end{bmatrix}[V_1\, V_2]^T$$

$$\approx U_1 S_1 V_1^T \quad \text{when } S_2 \approx 0,\ U_2 S_2 V_2^T \approx 0$$

Output normal realization

$$\underline{\underline{O_L = U_1}} \quad \text{and} \quad \underline{\underline{C_J = S_1 V_1^T}}$$

n = non zero singular values in S, i.e

$$S_1 = \begin{bmatrix} s_1 & & & 0 \\ & s_2 & & \\ & & \ddots & \\ 0 & & & s_n \end{bmatrix} n$$

d) Solve

$$H_{2|L} = O_L A C_J \Rightarrow H_{2|L} = U_1 A S_1 V_1^T$$

for A

$$\underline{\underline{A = U_1^T H_{2|L} V_1 S_1^{-1}}}$$

e) Yes

Proof

$$H_{2|L} = O_L A C_J = \overbrace{O_L A O_L^+}^{\hat{A}_L} \cdot \overbrace{O_L C_J}^{H_{1|L}} = \hat{A}_L H_{1|L}$$