University of South-Eastern Norway

Examination Information Page

Written examination

General information about the exam:

Subject code: IIA2217

Subject name: System Identification and Optimal estimation

Examination date: 24.05.2024

Examination time: 9.00-14.00

Total hours: 5

Responsible course manager: David Di Ruscio

Campus: Porsgrunn

Faculty: Technology, Natural Sciences and Maritime Sciences

No. of assignments: 5

No. of attachments: 0

No. of pages incl. information page and attachments: 6

Permitted aids:

Pen and paper

Comments:

Click or tap here to enter text.

Exam assignment:

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Spreadsheets ☒ Line sheets ☐

THE CANDIDATE MUST VERIFY THAT THE TASK SET IS COMPLETE ON THEIR OWN.

Task 1 (30%): Linear Regression and realization theory

a) Given a 1st order system

$$x_{k+1} = ax_k + bu_k + ke_k, \quad (1)$$
$$y_k = x_k + e_k, \quad (2)$$

where a, b and k are unknown parameters and e_k is white noise with covariance matrix $\mathrm{E}(e_k e_k^T) = \Delta$.

Answer the following:

- When may the model Eq. (1) and (2) be written as a linear regression model of the form

$$y_k = \varphi_k^T \theta_0 + e_k. \quad (3)$$

- Define the parameter vector θ_0 and the vector φ_k of regressors in this case.

b) Consider the linear regression model Eq. (3) and known data variables y_k and φ_k for known discrete time instants $k = 1, \ldots, N$.

Consider the following prediction error criterion

$$V_N(\theta) = \frac{1}{N}\sum_{k=1}^{N} \varepsilon_k^T \Lambda \varepsilon_k \quad (4)$$

where Λ is a specified and symmetric weighting matrix.

- Define the Prediction Error ε_k ?
- Find the Ordinary Least Squares (OLS) estimate, $\hat{\theta}_N$, of the true parameter vector θ_0.
- Does there exist an optimal weighting matrix Λ?

c) Orthogonal projections. Consider the linear equation

$$Y = OX + E, \quad (5)$$

where the matrices $Y \in \mathbb{R}^{m\times N}$ and $X \in \mathbb{R}^{n\times N}$ are known matrices. The matrix $E \in \mathbb{R}^{m\times N}$ is a matrix of noise and uncorrelated with X. We assume $N > n$.

- Define the mathematical definitions for the orthogonal projections Y/X and $YX^\perp$?
- Write down a two dimensional figure to illustrate the projections?
- Is $Y = Y/X + YX^\perp$ correct or not ?

d) Ordinary Least Squares regression

- Find an estimate of O in Eq. (5) ?
- Find a prediction $\bar{Y}$ of Y in Eq. (5) ?
- Is there an connection with the projection Y/X ?

e) Consider the linear equation

$$Y = XB + E, \tag{6}$$

where here the column vector $Y \in \mathbb{R}^N$ and the matrix $X \in \mathbb{R}^{N \times p}$ are known. $E \in \mathbb{R}^N$ is here a noise vector.

- Find an expression for the Ordinary Least Squares (OLS) regression estimate B_{OLS} of B ?

Task 2 (20%): Kalman Filter, Prediction Error Method

Given a linear system

$$x_{k+1} = Ax_k + Bu_k + v_k, \tag{7}$$
$$y_k = Dx_k + w_k, \tag{8}$$

where v_k and w_k are white process and measurements noise, respectively.

a) Formulate a Kalman filter on *apriori-aposteriori* form for the linear system in Eqs. (7)-(8), i.e. involving variables $\bar{x}_k$ and $\hat{x}_k$ and $\bar{y}_k$.

b) Formulate the *innovations formulation* of the Kalman filter.

c) The Kalman-filter on prediction form is often used in connection with Prediction error Methods (PEM) for system identification.

- Write down the Kalman filter on prediction form for the system in Eqs. (7) and (8) ?
- Define the Prediction Error (PE) ε_k ?
- Assume that we want to use PEM to find the parameters in a single input single output system with $n = 2$ states. Write down the structure of a 2nd order model on canonical form and define the parameter vector θ ?

d) Given a non-linear system

$$x_{k+1} = f(x_k, u_k) + v_k \tag{9}$$
$$y_k = g(x_k) + w_k \tag{10}$$

where v_k and w_k are discrete white process noise and discrete white measurements noise, respectively.

Formulate the Kalman filter on apriori-aposteriori form for the non-linear system model in Eqs. (9) and (10).

Task 3 (30%): Subspace System Identification: Combined Open and Closed Loop Systems

Consider the discrete time model on innovations form, ie.

$$x_{k+1} = Ax_k + Bu_k + Ce_k, \tag{11}$$
$$y_k = Dx_k + Eu_k + Fe_k \tag{12}$$

where e_k is white noise with unit covariance matrix, i.e., $\mathrm{E}(e_k e_k^T) = I$ and where the following output and input data matrices are known

$$Y = \begin{bmatrix} y_0^T \\ y_1^T \\ \vdots \\ y_{N-1} \end{bmatrix} \in \mathbb{R}^{N\times m}, \quad U = \begin{bmatrix} u_0^T \\ u_1^T \\ \vdots \\ u_{N-1} \end{bmatrix} \in \mathbb{R}^{N\times r}. \tag{13}$$

Based on the model in Eqs. (11) and (12) and with known data as given in (13) we can develop the following matrix equations

$$Y_{J|L} = O_L X_J + H_L^d U_{J|L+g-1} + H_L^s E_{J|L}, \tag{14}$$
$$Y_{J+1|L} = \tilde{A}_L Y_{J|L} + \tilde{B}_L U_{J|L+g} + \tilde{C}_L E_{J|L+1}, \tag{15}$$

where $J \geq 1$ and $L \geq 1$ are user specified positive integer numbers. We assume open loop data and that

$$E_{J|L+1} / \begin{bmatrix} U_{J|L+g} \\ U_{0|J} \\ Y_{0|J} \end{bmatrix} = 0 \tag{16}$$

a) Write down the structure of the matrices in the matrix Eqs. (14) and (15), with parameters $N = 10$, $L = 2$, $J = 2$ and $g = 0$.

b) By using (13) and Eqs. (14) and (15) we may formulate the equations

$$Z_{J|L} = O_L X_J^a \tag{17}$$

and

$$Z_{J+1|L} = \tilde{A}_L Z_{J|L} \tag{18}$$

Find expressions for the data matrices $Z_{J|L}$ and $Z_{J+1|L}$ for the following case:

- a general combined deterministic and stochastic system.

Remark: define the projections which is involved in the expressions for $Z_{J+1|L}$ and $Z_{J|L}$ in this case.

c) Show how

- the system order, n
- the extended observability matrix O_L
- the system matrices A and D

can be estimated.

d) Consider now the following Kalman filter on innovations form

$$x_{k+1} = Ax_k + Bu_k + K\varepsilon_k, \tag{19}$$
$$y_k = Dx_k + Eu_k + \varepsilon_k \tag{20}$$

What is the relationship between the Kalman filter on innovations form in Eqs. (19) and (20) and the innovations formulation in Eqs. (11) and (12)?

e) Consider that the known input and output data as given in (13) are collected in closed loop, i.e., we assume that there is feedback in the known data.

From Eq. (12) with $E = 0$, or Eq. (14) with $L = 1$ and $g = 0$, we may define the matrix Equation

$$Y_{J|1} = \overbrace{DX_{J|1}}^{y^d_{J|1}} + \overbrace{FE_{J|1}}^{\varepsilon_{J|1}} \tag{21}$$

Use that

$$E_{J|1} / \begin{bmatrix} U_{0|J} \\ Y_{0|J} \end{bmatrix} = 0 \tag{22}$$

and

$$X_{J|1} / \begin{bmatrix} U_{0|J} \\ Y_{0|J} \end{bmatrix} = X_{J|1} \quad \text{when } J \to \infty \tag{23}$$

We want to split/filter the output data $Y_{J|1}$ in Eq. (21) into signal part $y^d_{J|1}$ and a noise part $\varepsilon_{J|1}$:

- Define the signal term

$$y^d_{J|1} = DX_{J|1} = ? \tag{24}$$

- Define the noise term

$$\varepsilon_{J|1} = FE_{J|1} = ? \tag{25}$$

Task 4 (20%): Realization Theory

Assume known impulse response matrices

$$H_k = DA^{k-1}B \ \forall \ k = 1, \cdots, 5. \tag{26}$$

a) Write down the Hankel matrices $\mathbf{H}_{1|L}$ and $\mathbf{H}_{2|L}$ where you should use $L = 2$.

b) How are $\mathbf{H}_{1|L}$ and $\mathbf{H}_{2|L}$ related to O_L, C_J and A ? (Here O_L and C_J are the extended observability and controllability matrices, respectively.)

c) Show how you can find the system order, n, the extended observability matrix O_L and the extended controllability matrix C_J from a Singular Value decomposition (SVD).

d) Find a formula for calculating the system matrix A ?

e) Is

$$H_{2|L} = \tilde{A}_L H_{1|L} \tag{27}$$

correct or not ? (YES or NO answer)