OPTIMAL MODEL BASED CONTROL:
System Analysis and Design

Lecture notes

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Preface

This book contains lecture notes which are used in the Advanced Control Theory course which is held at the master study in Systems and Control Engineering at Faculty of Technology at University of South-Eastern Norway.

Some of the chapters is based on translated lecture notes in Norwegian. Hence, some of the theory also exists in Norwegian.

These lecture notes should give depth insight in Optimal Control of continuous as well as discrete time systems.

System theory, optimal control theory and estimation theory is central topics in the course. There also is one remarkable equation which comes up at diverse places in those topics, namely the Riccati Equation, after Count Jacopo Francesco Riccati and his paper from 1724.

In order to give an historical perspective we end this preliminary words by a verse written by Count Riccati:

> Since adolescence, the mind should be educated to treasure the most
> eminent of sciences and the finest of arts.
> I do not want to claim that every topic should be probed in detail.
> Following one's own talent and inclination, one should select at least
> one topic, and study it in depth. In the others, one should follow
> the example of the bee which sucks a drop of nectar from each flower...

This cite is from the *Opere of Count Jacopo Riccati ca. year 1676-1754.* See Bittanti, S. (1989).

Contents

Part I

SYSTEM THEORY

Chapter 1

Topics in Analysis of Linear Systems

1.1 Continuous time linear state space models

Definition 1.1 (Strictly proper linear state space model)
A continuous time, time invariant, *strictly proper* linear state space model is defined as follows

$$\dot{x} = Ax + Bu, \tag{1.1}$$
$$y = Dx, \tag{1.2}$$

where $u \in \mathbb{R}^r$ is the input vector, $x \in \mathbb{R}^n$ is the state vector and $y \in \mathbb{R}^m$ is the output vector. $x(t_0) = x_0$ is the initial state at the initial time t_0. The time invariant (constant) matrices A, B and D are of dimensions $n \times n$, $n \times r$ and $m \times n$, respectively.
$\triangle$

Definition 1.2 (Proper linear state space model)
The linear model in Definition 1.1 is only *proper* if there is a direct influence from the input vector u to the output vector y in the output equation, Eq. (1.2), i.e.

$$\dot{x} = Ax + Bu, \tag{1.3}$$
$$y = Dx + Eu, \tag{1.4}$$

where E is a $m \times r$ constant matrix.
$\triangle$

Equation (1.1) is referred to as the *state equation* and Equation (1.2) is referred to as the *output equation*. The *output equation* is some times referred to as the measurement equation or equation of measurements. The dimension n of the state vector x is referred to as the *system order*. The matrix A is referred to as the *state matrix*, the matrix B is referred to as the *input matrix* or also the *control input matrix*, and D, E is referred to as *output matrices*. Furthermore, the linear model, Equations (1.1) and (1.2), is defined to be *deterministic* if the input vector u is exactly known.

Definition 1.3 (Combined deterministic and stochastic model)
A continuous time, time invariant, combined deterministic and stochastic model is defined as follows

$$\dot{x} = Ax + Bu + Cv, \tag{1.5}$$
$$y = Dx + Eu + w, \tag{1.6}$$

where u is the known (deterministic) input vector, v is the stochastic (usually unknown) process noise vector and w is the stochastic measurements noise vector.
$\triangle$

Remark 1.1 *Note that an only proper state space model as defined in (1.3) and (1.4) can be expressed as the following strictly proper state space model*

$$\begin{bmatrix} \dot{x} \\ \dot{u} \end{bmatrix} = \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} + \begin{bmatrix} 0 \\ I \end{bmatrix} \dot{u} \tag{1.7}$$
$$y = \begin{bmatrix} D & E \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} \tag{1.8}$$

1.2 Solution to the continuous state equation

The state equation $\dot{x} = Ax + Bu$ have the following solution

$$x(t) = e^{A(t-t_0)}x(t_0) + \int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)d\tau. \tag{1.9}$$

The initial time is often assumed to be zero, i.e. $t_0 = 0$. The transition matrix Φ is defined as

$$\Phi(t, t_0) = e^{A(t-t_0)}. \tag{1.10}$$

The solution $x(t)$ given by Equation (1.9) can be written in terms of the transition matrix Φ as follows

$$x(t) = \Phi(t, t_0)x(t_0) + \int_{t_0}^{t} \Phi(t, \tau)Bu(\tau)d\tau. \tag{1.11}$$

A special case which is of particular practical importance in connection with discretization of continuous models is to consider the case were $u(\tau)$ is constant in the time interval $t_0 \leq \tau < t$. Hence we have that (1.11) can be written as

$$x(t) = e^{A(t-t_0)}x(t_0) + A^{-1}(e^{A(t-t_0)} - I)Bu(t_0) \tag{1.12}$$

when A is non-singular. This can be proved as follows

$$\begin{aligned} \textstyle\int_{t_0}^{t} \Phi(t,\tau)Bu(\tau)d\tau &= (\textstyle\int_{t_0}^{t} e^{A(t-\tau)}Bd\tau)u(\tau) = \left[-A^{-1}e^{A(t-\tau)} \right]_{t_0}^{t} Bu(t_0) \\ &= (-A^{-1} - (-A^{-1})e^{A(t-t_0)})Bu(t_0) = A^{-1}(e^{A(t-t_0)} - I)Bu(t_0) \end{aligned} \tag{1.13}$$

where we have used that $u(\tau) = u(t_0)$ in the time interval $t_0 \leq \tau < t$. The integral in (1.11) can also be solved for the case when A is singular. See exercise 19.1 and solution 19.1 for an example.

1.3 Discrete time linear state space models

For some linear systems the state, input, output and noise vectors are defined only at fixed time instants, say

$$t_k = k\Delta t, \tag{1.14}$$

where $k \geq 0$ is defined as the discrete time, usually the integer values

$$k = 0, 1, 2, ... \tag{1.15}$$

and Δt is the *sampling interval*, usually a constant time interval.

If an arbitrarily continuous vector signal $u(t)$ is sampled at the discrete time instants as specified above, then we have a sequence of vectors defined only at discrete time instants

$$u(t_k) = u(k\Delta t) \ \forall \ k = 0, 1, \dots \tag{1.16}$$

We will make the following shorthand notation

$$u_k \stackrel{\text{def}}{=} u(t_k) = u(k\Delta t). \tag{1.17}$$

In a Digital Control System (DCS) we frequently have that the input $u(t)$ to the process is applied periodically at time instants $t_k = k\Delta t$ and held constant within the period, i.e.

$$u(t) = u_k \ \forall \ k\Delta t \leq t < (k+1)\Delta t \text{ and } k = 0, 1, \dots \tag{1.18}$$

A discrete signal u_k can be converted to a stepwise constant continuous signal $u(t)$ as defined in (1.18) by using a *zero-order hold element*, i.e. a digital to analog converter.

In digital control systems a discrete input u_k to the process is usually computed by a digital controller. The digital (discrete) signal u_k must be converted to an analog (continuous) signal before being sent to the process (or final control element, such as e.g. a valve position). One of the most common digital to analog converters is the *zero-order hold element* which results in a signal $u(t)$ as described above in (1.18).

Another digital to analog converter is the *first-order hold element*. A first-order hold assumes that the signal changes linearly as predicted from e.g. the two recent samples u_{k-1} and u_k

Suppose now that the continuous output $y(t)$ from the process is observed also periodically at discrete time instants of time which, however, need not coincide in time with the time instants at which the inputs are adjusted. Define

$$y_k = y(k\Delta t + \Delta t') \text{ where } 0 \leq \Delta t' < \Delta t \text{ and } k = 0, 1, \dots \tag{1.19}$$

We will call $\Delta t'$ the displacement in time between the sampled variables u_k and y_k.

A discrete time state space model is presented in the following definition.

Definition 1.4 (Discrete time, proper state space model)
A discrete time, time invariant, proper state space model is defined as follows

$$x_{k+1} = Ax_k + Bu_k, \tag{1.20}$$
$$y_k = Dx_k + Eu_k, \tag{1.21}$$

where $u_k \in \mathbb{R}^r$ is the input vector, $y_k \in \mathbb{R}^m$ is the output vector and $x_k \in \mathbb{R}^n$ is the state vector. A is the state transition matrix and E is the direct feed-through matrix. x_0 is the initial time state vector. x_0 is usually specified.
△

Note that the discrete time system may have a direct feed-through term $E \neq 0$ even if the underlying continuous time system has not. The reason for this is e.g. the presence of a displacement $\Delta t'$ in time between the input u_k and the output y_k.

Hence, a discrete version of a continuous model $\dot{x} = A_c x + B_c u$ and $y = D_c x$ is given by (1.20) and (1.21) with the discrete model matrices

$$\begin{array}{ll} A = e^{A_c \Delta t} & B = \int_0^{\Delta t} e^{A_c \tau} B_c d\tau \\ D = D_c e^{A_c \Delta t'} & E = D_c \int_0^{\Delta t'} e^{A_c \tau} d\tau \end{array}$$

and where Δt is the sampling time and $\Delta t'$ is the displacement between the input and the output. A common special case is to assume that the displacement $\Delta t' = 0$. In this case we have that $D = D_c$ and $E = 0$.

A linear or linearized system which is influenced by process noise v_k and measurements noise w_k can be described as in the following definition.

Definition 1.5 (Discrete combined deterministic and stochastic model)
A discrete time, time invariant, combined deterministic and stochastic model is defined as follows

$$x_{k+1} = Ax_k + Bu_k + Cv_k, \tag{1.22}$$
$$y_k = Dx_k + Eu_k + w_k, \tag{1.23}$$

where u_k is the input vector, v_k is the process noise vector and w_k is the measurements noise vector.
△

Remark 1.2 *Note that the only proper state space model, as defined in (1.20) and (1.21), can be expressed as the following strictly proper state space model*

$$\begin{bmatrix} x_{k+1} \\ u_{k+1} \end{bmatrix} = \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix} \overbrace{\begin{bmatrix} x_k \\ u_k \end{bmatrix}}^{\tilde{x}_k} + \begin{bmatrix} 0 \\ I \end{bmatrix} u_{k+1} \tag{1.24}$$
$$y = \begin{bmatrix} D & E \end{bmatrix} \begin{bmatrix} x_k \\ u_k \end{bmatrix} \tag{1.25}$$

where the initial time state vector is given by

$$\tilde{x}_0 = \begin{bmatrix} x_0 \\ u_0 \end{bmatrix} \tag{1.26}$$

We have here assumed that the initial time is $k = 0$.

Some alternative methods for reformulating an *only proper* state space model into a *strictly proper* state space model are discussed and presented in Exercises **??** - **??** and Solutions **??** - **??**.

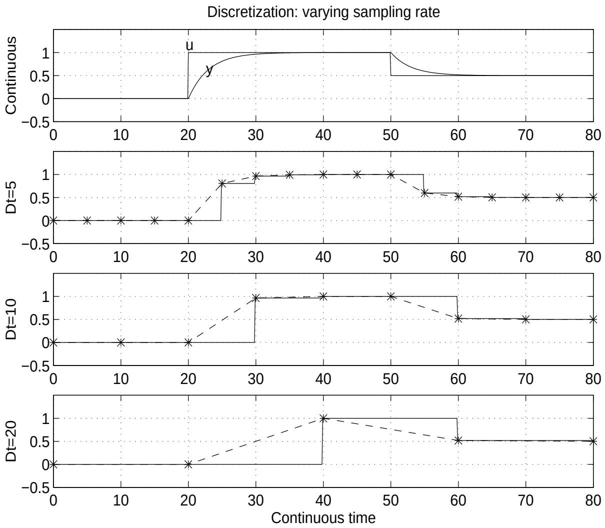

Figure 1.1: A 1.st order continuous model ($\dot{x} = -\frac{1}{3}x + u$, $y = \frac{1}{3}x$) excited with a unit input step response is discretized with varying sampling rate.

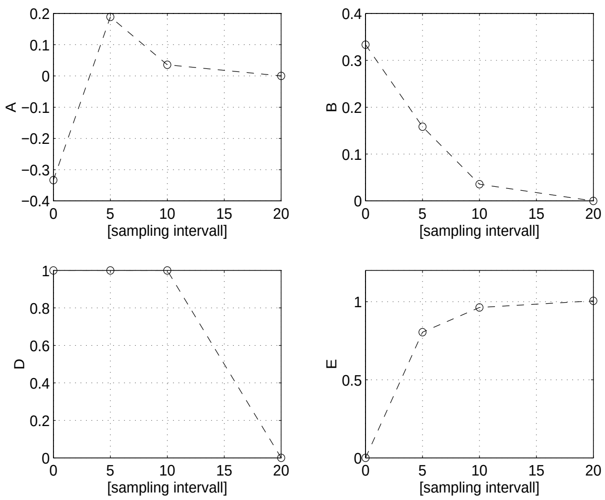

Figure 1.2: A 1.st order continuous model ($\dot{x} = -\frac{1}{3}x + u$, $y = \frac{1}{3}x$) exited with a unit input step response is discretized with varying sampling rate. The discrete state space model parameters are illustrated as a function of the sampling rate. The continuous model is strictly proper ($E = 0$).

Example 1.1 (Effect of sampling a continuous SS model)
Consider a continuous time, strictly proper state space model given by

$$\dot{x} = -\frac{1}{3}x + u, \quad y = \frac{1}{3}x. \tag{1.27}$$

The continuous response after an input experiment is illustrated in Figure (1.1). The continuous time model is simulated from time $t_0 = 0$ *to* $t_1 = 79.9$ *by using the Matlab Control System Toolbox function lsim.m. The continuous time instants is generated by* $t = 0 : 0.1 : 79.9$ *which results in* 800 *time instants with a time increment (sampling time) of* 0.1.

The data which results from the simulation of the continuous time model is sampled with varying sampling interval of $\Delta t = 5$, $\Delta t = 10$ *and* $\Delta t = 20$. *The discrete time instants are also illustrated in Figure (1.1).*

It can be shown, by e.g. using a system identification method, that the discrete time instants are exactly given by a proper state space model of the form

$$x_{k+1} = Ax_k + Bu_k \tag{1.28}$$

$$y_k = Dx_k + Eu_k \tag{1.29}$$

where the discrete state space model parameters are as illustrated in Figure (1.2) and presented in the table below.

Δt	0	5	10	20
A	$-\frac{1}{3}$	0.1889	0.0357	0
B	1	0.1584	0.0356	0
D	$\frac{1}{3}$	1	1	0
E	0	0.8047	0.9631	1.0053

(1.30)

The discrete model parameters shown in Figure 1.2.

△

Example 1.1 illustrates the fact that sampling a *strictly proper* continuous state space model may give rise to a discrete time state space model which is *only proper*, i.e. a state space model characterized with a direct feed-through term Eu_k from the input u_k to the output y_k.

The reason for this is usually the presence of some kind of displacement in time between the signals. E.g., a small displacement in time between the input u_k and the output y_k.

Remark 1.3 *Consider a continuous model* $\dot{x} = A_c x + B_c u$ *and that the input is constant over time (sampling) intervals of size* $\Delta t > 0$, *i.e.,* $u(t)$ *is constant for* $t_k \leq t < t_k + \Delta t$. *An exact discrete time model can then be derived from (1.12) and is given by*

$$x_{k+1} = Ax_k + Bu_k \tag{1.31}$$

where

$$A = e^{A_c \Delta t}, \tag{1.32}$$

$$B = A_c^{-1}(e^{A_c \Delta t} - I)B_c. \tag{1.33}$$

1.4 Controllability

Definition 1.6 (Controllability)
The linear system, Equation (1.1), is said to be completely (state) controllable if for any initial state vector $x_0 = x(t_0)$ there exist a finite time t_f and a control vector $u(t)$ for the time interval $t_0 \leq t \leq t_f$ which moves the state vector to a prescribed final state vector $x_f = x(t_f)$.
$\triangle$

It exists several criteria for controllability which gives us a (yes or no) answer to whether a linear system, defined by the pair (A, B), is controllable or uncontrollable.

Theorem 1.4.1 (Controllability matrix)
The pair (A, B) is controllable if and only if the *controllability matrix*

$$C_n = \begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix} \quad \in \ \mathbb{R}^{n \times n \cdot r}, \tag{1.34}$$

has rank n, i.e. $\mathrm{rank}(C_n) = n$.

If $\mathrm{rank}(B) = r_B \geq 1$, then, this condition reduces to

$$C_{n-r_B+1} = \begin{bmatrix} B & AB & A^2B & \cdots & A^{n-r_B}B \end{bmatrix} \quad \in \ \mathbb{R}^{n \times (n-r_B+1) \cdot r}, \tag{1.35}$$

where we have assumed that $n - r_B + 1 > 0$. The pair (A, B) is controllable if and only if the reduced controllability matrix C_{n-r_B+1} have rank n.
$\triangle$

Theorem 1.4.1 is valid for both continuous time and discrete time models. Unfortunately, this theorem may give a wrong answer, since the computations of the controllability matrix (C_n) may be related to great errors, because of subtractive cancelations in evaluating the powers of A. For multi input systems, $r > 1$ and $rank(B) = r_B > 1$, Equation (1.35) is recommended (if Theorem 1.4.1 is to be used), because powers of A only up to A^{n-r_B} has to be computed. The rank test of the controllability matrix works well on small systems which can be solved exactly by hand, but it may lead to a very poor algorithm when used as the basis of machine software.

The MATLAB Control System Toolbox function **ctrb** can be used to form the controllability matrix C_n, i.e. $C_n = \mathrm{ctrb}(A, B)$.

Example 1.2 (Controllability)
Given a system described by

$$A = \begin{bmatrix} 1 & \delta \\ 0 & \delta \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ \delta \end{bmatrix}. \tag{1.36}$$

The controllability matrix for this system is given by

$$C_2 = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 1 & 1+\delta^2 \\ \delta & \delta^2 \end{bmatrix}. \tag{1.37}$$

The system is controllable if $\delta \neq 0$ because rank$(C_2) = 2$ in this case.

But if a computer with machine precision eps is used to compute C_2, then we will get

$$C_2 = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ \delta & \delta^2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \tag{1.38}$$

when $\delta < \sqrt{eps}$. The reason for this is that $\delta = \delta^2 = 0$ in this case. Note that rank$(C_2) = 1$ in this last case. The computer based controllability test says that the system is not controllable even if it is.

1.4.1 Continuous time controllability Gramian

Theorem 1.4.2 (Continuous controllability Gramian)
Assume the linear continuous time model. The pair (A, B) is controllable if and only if the $n \times n$ *controllability Gramian*

$$W_c(t) = \int_0^t e^{A\tau} B B^T e^{A^T \tau} d\tau \quad \in \quad \mathbb{R}^{n \times n}. \tag{1.39}$$

is positive definite for some $t > 0$. W_c is positive definite if and only if rank$(W_c) = n$.

$\triangle$

If A is a stable matrix, then for $t \to \infty$, the continuous infinite time controllability Gramian satisfy the Lyapunov matrix equation

$$A W_c + W_c A^T = -B B^T. \tag{1.40}$$

The Lyapunov equation is linear in the elements w_{ij} of the Controllability Gramian W_c. There exist numerically stable and efficient algorithms for solving the linear matrix Lyapunov equation. Hence, it is a better method than the rank test, Theorem (1.4.1), for controllability analysis. The MATLAB Control System Toolbox function **gram** can be used to compute the continuous time controllability Gramian W_c. The function **gram** solves the Lyapunov equation (1.40) for W_c. **gram** works only for stable systems. A method for computing W_c which also works for unstable systems is presented below.

Proof of Equation (1.40)
Substitute Equation (1.39) into the left hand side of Equation (1.40). We have

$$\begin{aligned} A W_c + W_c A^T &= \int_0^t A e^{A\tau} B B^T e^{A^T \tau} d\tau + \int_0^t e^{A\tau} B B^T e^{A^T \tau} A^T d\tau \\ &= \int_0^t \frac{d}{d\tau} (e^{A\tau} B B^T e^{A^T \tau}) d\tau \\ &= [e^{A\tau} B B^T e^{A^T \tau}]_0^t = e^{At} B B^T e^{A^T t} - B B^T. \end{aligned} \tag{1.41}$$

which is identical to the Lyapunov matrix Equation (1.40) when A is stable and $t \to \infty$. **QED**

If A is unstable, the Gramian Equation (1.39), can be solved directly for some finite t. Hence, in the general case the Gramian can be solved as follows. Compute the following matrix exponential

$$\begin{bmatrix} E_{11} & E_{12} \\ 0 & E_{22} \end{bmatrix} = e^{\begin{bmatrix} -A & BB^T \\ 0 & A^T \end{bmatrix} t}, \tag{1.42}$$

The Gramian is then given by

$$W_c(t) = E_{22}^T E_{12}. \tag{1.43}$$

We shall however note that a simple method for computing the controllability Gramian $W_c(t)$ for a specified finite time t, can be done by solving the Lyapunov matrix equation

$$AW_c(t) + W_c(t)A^T = e^{At}BB^T e^{A^T t} - BB^T \tag{1.44}$$

for $W_c(t)$. This follows from Equation 1.41.

1.4.2 Control input for specified state

The input which achieves the state $x(t_1)$ is given by

$$u(t) = -B^T e^{A^T(t_1-t)} W_c^{-1}(t_1 - t_0)(e^{A(t_1-t_0)}x(t_0) - x(t_1)). \tag{1.45}$$

where $W_c(t)$ is defined in (1.39). This expression can be derived from linear quadratic optimal control theory. However, a more direct proof is given in the following.

Proof: From Equation (1.9) with $t = t_1$ we have

$$x(t_1) = e^{A(t_1-t_0)}x(t_0) + \int_{t_0}^{t_1} e^{A(t_1-\tau)}Bu(\tau)d\tau. \tag{1.46}$$

We will below show that the control input defined by (1.45) satisfy (1.46). Substituting $u(\tau)$ given by (1.45) into (1.46) gives

$$x(t_1) = e^{A(t_1-t_0)}x(t_0) - \underbrace{\int_{t_0}^{t_1} e^{A(t_1-\tau)}BB^T e^{A^T(t_1-\tau)}d\tau}_{W_c(t_1-t_0)} W_c^{-1}(t_1-\tau)(e^{A(t_1-t_0)}x(t_0) - x(t_1)) \ . \tag{1.47}$$

The integral which is under-braced can be shown to be identical to the Gramian $W_c(t_1)$. This can be shown by changing the integration variable from τ to e.g. s. Defining $s = t_1 - \tau$ gives $ds = -d\tau$ and integration from $s_0 = t_1 - 0 = t_1$ to $s_1 = t_1 - t_1 = 0$ gives.

$$\int_{t_0}^{t_1} e^{A(t_1-\tau)}BB^T e^{A^T(t_1-\tau)}d\tau = -\int_{t_1-t_0}^{0} e^{As}BB^T e^{A^T s}ds = W_c(t_1 - t_0), \tag{1.48}$$

which is identical to the controllability Gramian (1.39). Finally from (1.47) we have

$$x(t_1) = e^{A(t_1-t_0)}x(t_0) - W_c(t_1-t_0)W_c^{-1}(t_1-t_0)(e^{A(t_1-t_0)}x(t_0) - x(t_1)).$$
$$\Downarrow$$
$$x(t_1) = e^{A(t_1-t_0)}x(t_0) - (e^{A(t_1-t_0)}x(t_0) - x(t_1)). \tag{1.49}$$
$$\Downarrow$$
$$x(t_1) = x(t_1).$$

QED.

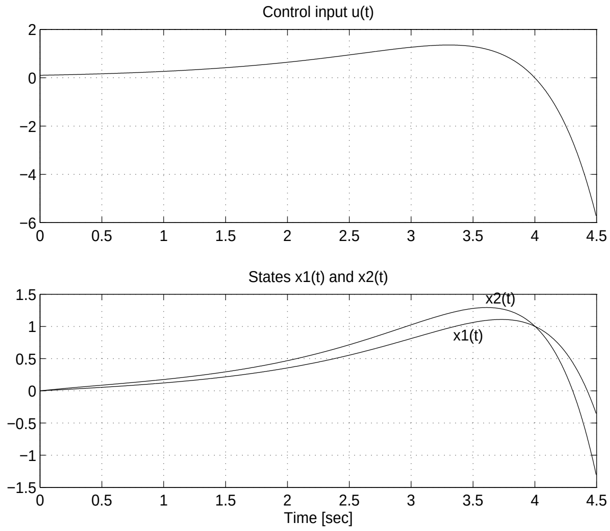

Figure 1.3: Illustration of state controllability. It was specified that the states at time $t = 4$ should be $x_1 = x_2 = 1$. Se Example 1.3 for details.

Example 1.3 *State controllability*
Consider the system

$$\dot{x} = \begin{bmatrix} -1 & 0.1 \\ 0.2 & -2 \end{bmatrix} x + \begin{bmatrix} 1 \\ 2 \end{bmatrix} u, \quad x(t_0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{1.50}$$

From the definition of state controllability we have that it exist a control signal $u(t)$ *which gives a final state vector* $x(t_1)$*.*

Assume that we want the state at time $t_1 = 4$ *to be*

$$x(t_1 = 4) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \tag{1.51}$$

Using (1.45) we get to input signal

$$u(t) = -\begin{bmatrix} 1 & 2 \end{bmatrix} e^{A^T(4-t)} \begin{bmatrix} -4.925 \\ 2.466 \end{bmatrix}. \tag{1.52}$$

This result is illustrated in Figure 1.3. Figure 1.3 shows that the states actually hit the target $x_1 = x_2 = 1$*. However, from Equation (1.45) we have that the input is unstable for* $t > t_1 = 4$ *when* A *is stable.*

1.4.3 Discrete time controllability Gramian

The discrete time equivalent to the Controllability Gramian theorem is as follows

Theorem 1.4.3 (Controllability Gramian)
Assume the linear discrete time model. The pair (A, B) is controllable if and only if the $n \times n$ discrete time *controllability Gramian*

$$W_c = \sum_{i=1}^{N} A^{(i-1)} B B^T A^{(i-1)T} \;\in\; \mathbb{R}^{n\times n}. \tag{1.53}$$

is positive definite for $N > n$. Same as $\mathrm{rank}(W_c) = n$.

$\triangle$

If A is a stable matrix, then for $N \to \infty$, the discrete infinite time controllability Gramian satisfy the discrete Lyapunov matrix equation

$$AW_cA^T - W_c = -BB^T. \tag{1.54}$$

The Lyapunov equation is linear in the elements w_{ij} of the Controllability Gramian W_c. This is a better method than the rank test, Theorem (1.4.1), for controllability analysis.

Note also that the discrete time controllability Gramian satisfy

$$W_c = C_N C_N^T, \tag{1.55}$$

where C_N is the extended controllability matrix. This gives immediately the link between the controllability matrix and the discrete controllability Gramian.

1.5 Steady state controllability

Consider a stable state space model

$$\dot{x} = Ax + Bu, \tag{1.56}$$
$$y = Dx + Eu. \tag{1.57}$$

In order to analyze the system in steady state the system must be stable, i.e A has all eigenvalues strictly in the left hand part of the complex plane.

We will in the following discuss perfect control and controllability. The transfer function model is then

$$y(s) = (D(sI - A)^{-1}B + E)u(s) \tag{1.58}$$

where s is the Laplace operator. In steady state we have $s = 0$. The continuous *proper* linear state space model is in steady state described by

$$x^s = -A^{-1}x^s + Bu^s, \tag{1.59}$$
$$y^s = (-DA^{-1}B + E)u^s. \tag{1.60}$$

where x^s, u^s and y^s are steady state vectors. Introduce the steady state gain matrix from the inputs u to the outputs y, i.e.

$$H^d = -DA^{-1}B + E. \tag{1.61}$$

Theorem 1.5.1 (Steady state output controllability)
If the system matrix A is non-singular, i.e. if A^{-1} exist, then the system is completely steady state output controllable, if and the steady state gain matrix $H^d = -DA^{-1}B + E$ is non-singular.
$\triangle$

This can be proved as follows. Assume that we want to force the output y to a prescribed set-point y^s in steady state by some control input vector u^s. It is immediately shown from the above that u^s is defined if and only if H^d is invertible, i.e. $u^s = (H^d)^{-1}y^s$.

1.6 Observability

Theorem 1.6.1 (Observability matrix)
Define the *observability matrix*

$$O_i = \begin{bmatrix} D \\ DA \\ DA^2 \\ \vdots \\ DA^{i-1} \end{bmatrix} \quad \in \ \mathbb{R}^{mi \times n}, \tag{1.62}$$

The pair (D, A) is observable if and only if the *observability matrix* O_i has rank n, i.e. $\text{rank}(O_n) = n$.

If $\text{rank}(D) = r_D \geq 1$ and $n - r_D + 1 > 0$, then we have that the pair (D, A) is observable if and only if the reduced observability matrix O_{n-r_D+1} have rank n.
$\triangle$

Theorem 1.6.2 (Continuous observability Gramian)
Consider the linear continuous time model. The pair (D, A) is observable if and only if the $n \times n$ *observability Gramian*

$$W_o(t) = \int_0^t e^{A^T\tau} D^T D e^{A\tau} d\tau \quad \in \ \mathbb{R}^{n \times n}. \tag{1.63}$$

is positive definite for some $t > 0$. W_o is positive definite if and only if $\text{rank}(W_o) = n$.

If A is a stable matrix, then for $t \to \infty$, the continuous infinite time observability Gramian satisfy the Lyapunov matrix equation

$$A^T W_o + W_o A = -D^T D. \tag{1.64}$$

$\triangle$

Theorem 1.6.3 (Discrete observability Gramian)
Consider the linear discrete time model. The pair (D, A) is observable if and only if the $n \times n$ discrete time *observability Gramian*

$$W_o = \sum_{i=1}^{N} A^{(i-1)T} D^T D A^{(i-1)} \;\in\; \mathbb{R}^{n \times n}. \tag{1.65}$$

is positive definite for $N > n$. Same as $\text{rank}(W_o) = n$.

If A is a stable matrix, then for $N \to \infty$, the discrete infinite time observability Gramian satisfy the discrete Lyapunov matrix equation

$$A^T W_o A - W_o = -D^T D. \tag{1.66}$$

In the general case the Gramian can be solved as follows. Compute the following matrix exponential

$$\begin{bmatrix} E_{11} & E_{12} \\ 0 & E_{22} \end{bmatrix} = e^{\begin{bmatrix} -A^T & D^T D \\ 0 & A \end{bmatrix} t}, \tag{1.67}$$

for some specified time $t > 0$. The observability Gramian is then given by

$$W_o(t) = E_{22}^T E_{12}. \tag{1.68}$$

$\triangle$

Note also that the discrete time observability Gramian satisfy

$$W_o = O_N^T O_N, \tag{1.69}$$

where O_N is the extended observability matrix for the pair (D, A). This gives immediately the link between the observability matrix and the discrete observability Gramian.

1.7 More on observability and controllability

Remark 1.4 (Diagonal form and observability and controllability)
Consider a state space model $\dot{x} = Ax + Bu$ and $y = Dx + Eu$ and its diagonal canonical form

$$\dot{z} = \Lambda z + M^{-1} B u \tag{1.70}$$

$$y = DMz + Eu \tag{1.71}$$

where Λ is a diagonal matrix with the eigenvalues $\lambda_i \;\forall\; i = 1, \ldots n$ of A on the diagonal and $M = \begin{bmatrix} m_1 & \cdots & m_n \end{bmatrix}$ is the corresponding eigenvector matrix. Note the relationship $Am_i = \lambda_i m_i$ between the ith eigenvalue and eigenvector.

The system is observable if no columns in the matrix DM is identically equal to zero. Furthermore, the system is controllable if no rows in the matrix $M^{-1}B$ is identically equal to zero.

Note that the controllability and observability tests is existence tests. They says nothing about the degree of controllability and observability. This is an important limitation.

1.8 Zeroes in multivariable linear systems

Zeros are usually and numerically preferred, computed from a state space realization of the system. The method is illustrated in the following.

The Laplace transform of the continuous time *proper* state space model is given by

$$sx(s) = Ax(s) + Bu(s), \tag{1.72}$$
$$y(s) = Dx(s) + Eu(s). \tag{1.73}$$

This system of equations can be written i matrix form as follows

$$\left(\begin{bmatrix} sI & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} A & B \\ D & E \end{bmatrix} \right) \begin{bmatrix} x \\ u \end{bmatrix} = \begin{bmatrix} 0 \\ -y \end{bmatrix}. \tag{1.74}$$

The zeroes are the values $s = s_0$ for which the matrix

$$sI_g - S = s \overbrace{\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}}^{I_g} - \overbrace{\begin{bmatrix} A & B \\ D & E \end{bmatrix}}^{S}, \tag{1.75}$$

loses rank. If s_0 is a zero frequency, then the matrix (1.75) will lose rank at $s = s_0$, and there will exist a vector

$$m_0 = \begin{bmatrix} x_0 \\ u_0 \end{bmatrix}, \tag{1.76}$$

such that

$$(s_0 I_g - S) \begin{bmatrix} x_0 \\ u_0 \end{bmatrix} = 0. \tag{1.77}$$

The zeroes are then computed as the finite generalized eigenvalues of the following generalized eigenvector/eigenvalue problem

$$Sm_0 = s_0 I_g m_0. \tag{1.78}$$

This is preferred for numerical calculations. Note that if $I_g = I$ this reduces to the conventional eigenvector/eigenvalue problem.

Note that the zeros can be calculated as the roots of the characteristic equation (for the generalized eigenvalue problem), i.e.,

$$\rho(s) = \det(s_0 I_g - S) = 0 \tag{1.79}$$

This method may be suitable for hand calculations of some simple systems, i.e., for systems which lead to an S matrix of at most dimension 4×4. The roots can, in general, be computed analytically in this case.

Note that the zero frequency s_0 results in zero output $y = 0$ for some non-zero input u_0 and initial value x_0. In other terms this means that an input

$$u = u_0 e^{s_0 t}, \tag{1.80}$$

results in an output $y \equiv 0$ for some initial state vector x_0.

Note also that zeroes in MIMO systems often are called *transmission zeroes*. The zeroes are generally different from the zeroes of the elements in the transfer matrix $H(s) = D(sI - A)^{-1}B + E$.

Example 1.4 (Transmission zeroes)
Given a continuous linear two-input and two-output (MIMO) system with system matrices

$$A = \begin{bmatrix} -\frac{1}{2} & 0 \\ 0 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad E = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \tag{1.81}$$

The generalized eigenvalue problem can be solved in MATLAB as $[\underline{m}_0, \underline{s}_0] = eig(S, I_g)$ *where* $\underline{s}_0$ *is a vector with the transmission zeroes and* $\underline{m}_0$ *is a vector with generalized eigenvectors satisfying* $S\underline{m}_0 = I_g \underline{m}_0 \underline{s}_0$.

There are two finite zeroes of this generalized eigenvalue problem, $s_0^1 = -\frac{1}{2}$ *and* $s_0^2 = 2$. *Hence, the system has a zero in the left half plane. The system is non-minimum-phase.*

Chapter 2

Multivariable Frequency Analysis

2.1 Stabilizability and detectability

Definisjon 2.1 (Controllability)
A system $\dot{x} = Ax + Bu$ is controllable if there exist a control vector $u(t)$ (defined over a finite time interval $t_0 \leq t \leq t_1$) which brings the system state vector $x(t)$ from an arbitrary initial state $x(t_0)$ to an arbitrary final state $x(t_1)$ within the final time interval.

Definisjon 2.2 (Stabilizability)
A system given by the matrix pair (A, B) is **stabilizable** *if all unstable modes (eigenvalues or poles) are controllable.*

Note that stabilizability is a weaker demand than controllability. In a stabilizable system there may be uncontrollable states but those states must be stable. Often there dos not matter if some states are uncontrollable, but it make sense to demand the system to be stable.

Definisjon 2.3 (Observability)
A system is observable if it by the knowledge of the system output measurements vector y and the input vector u within a finite time interval ($t_0 \leq t \leq t_1$) is possible to compute all elements (variables) in the state vector $x(t)$.

Definisjon 2.4 (Detectability)
A linear system given by the matrix pair (D, A) is **detectable** *if all unstable modes in the system (i.e. eigenvalues ore poles in the system) is observable.*

Remark that detectability is a weaker demand than observability. A detectable system may have un observable states, but those un observable states must be stable for the system to be detectable. The above definitions are central in connection with existence analysis of the solution to the linear quadratic optimal control problem as well as the dual linear optimal estimation problem, i.e. the Kalman filter. If the system matrix, A, can be diagonalized, i.e. if there exists an eigenvalue matrix M and a diagonal eigenvalue matrix Λ such that $\Lambda = M^{-1}AM$ ore equivalently $A = M\Lambda M^{-1}$, then stabilizability and detectability analysis can be performed by viewing rows in $M^{-1}B$ and columns in DM, respectively. The system is stabilizable if no rows in $M^{-1}B$ witch belongs to unstable eigenvalues (positive eigenvalues), are identically equal to the zero vector. In the same way, the system is detectable if no columns in DM, which belongs to unstable eigenvalues, are identically equal to the zero vector.

In connection with linear dynamic systems we often speak of the modes of the system. the modes of a realization (A, B, D) is described by the eigenvalues of the system matrix A. In connection with this we also have modal analysis and modal control. Modal analysis of a system (A, B, D) is performed on the equivalent diagonalized system $(\Lambda, M^{-1}B, DM)$ where $\Lambda = M^{-1}AM$ is a diagonal eigenvalue matrix, if the eigenvector matrix M is non-singular (invertible). Model control means to find the controller such that the closed loop system gets prescribed modes (ore eigenvalues).

2.2 System poles and related definitions

It is important to remark that the poles of a linear dynamic system usually are computed numerically by computing the eigenvalues of the system matrix A in the linear state space model. This state space model should (but not necessary) be a minimal realization in order to get as few poles as possible.

Definisjon 2.5 (Poles from state space model)
The poles of a system given by the state space model $\dot{x} = Ax + Bu$, $y = Dx + Eu$ is given by the eigenvalues $\lambda_i \ \forall \ i = 1, \ldots, n$ to the system matrix A. The pole polynomial ore the characteristic polynomial for A is defined as

$$\pi(s) = \det(sI - A) = s^n + a_n s^{n-1} + \cdots + a_2 s + a_1 = \prod_{i=1}^{n}(s - s_i) \tag{2.1}$$

where $s_i = \lambda_i \ \forall \ i = 1, \ldots, n$ is the poles of the system. An alternative is

$$\pi(\lambda) = \det(\lambda I - A) = \lambda^n + a_n \lambda^{n-1} + \cdots + a_2 \lambda + a_1 = \prod_{i=1}^{n}(\lambda - \lambda_i) \tag{2.2}$$

where $\lambda_i \ \forall \ i = 1, \ldots, n$ is the poles of the system. Hence, the poles are given by the roots of the characteristic equation, i.e., $\pi(s) = \det(sI - A) = 0$.

We define, n, as the order of the dynamic system, if the state space model is a minimal realization, i.e., so that the state space model does not contain unnecessary states which are not controllable and observable.

The definition is valid if the state space model is a minimal realization or not. If the state space model is not a minimal realization, then we will have poles that describes redundant states which is uncontrollable and unobservable. Note that a minimal realization can be computed in MATLAB by the function minreal(A, B, D, E).

Definisjon 2.6 (Minimal realization)
A state space realization (A, B, D) is minimal if and only if the pair (A, B) is controllable and the pair (D, A) is observable.

If (A, B, D) is a minimal realization then the system matrix A has least possible dimension, i.e., the system order, n, in a minimal realization is minimal.

If the transfer matrix $H(s)$ of a system is given, then this model can be transformed to a state space model and the system poles can then be computed from the eigenvalues of the system matrix A. However, in some cases it may make sense to compute the poles directly from the transfer function model $H(s)$ directly. One advantage is that the calculations is easy to perform by hand. the calculations will also directly give the poles corresponding to a minimal state space realization.

Definisjon 2.7 (Poles from transfer matrix model $H(s)$)
The pole polynomial $\pi(s)$ is given by the smallest common denominator for all under determinants, which is not identically zero, of all orders of the system transfer matrix $H(s)$. The pole polynomial is then given by

$$\pi(s) = \prod_{i=1}^{n}(s - s_i) \tag{2.3}$$

where $s_i \ \forall \ i = 1, \ldots, n$ is the system poles.

The poles of the system is then given by the roots of the pole polynomial $\pi(s)$.

One weakness with this definition is that it gives no reliable method to be implemented in a computer. The problem is to find the roots of polynomials because this is numerically difficult. The problem is badly conditioned for numerically computations in a computer. the most robust and reliable method of computing poles in a computer is to do the calcluations by computing the eigenvalues of the A matrix.

2.3 Poles and stability

Teorem 2.3.1 (Stability in linear dynamic systems)
A linear dynamic system $\dot{x} = Ax + Bu$ is stable if and only if all poles (ore eigenvalues) $\lambda_i \ \forall \ i = 1, \ldots, n$ to the system matrix A is located in the left half part of the complex plane. This is equivalent with that the real part of the poles is negative, i.e., $\mathbb{R}e\{\lambda_i(A)\} < 0$.

2.4 Zeroes in multivariable systems

An important meaning of a zero, say s_0, is that the effect of a control input, $u(s_0) \neq 0$, on the system is such that the output is zero, i.e. $y(s_0) = H(s_0)u(s_0) = 0$.

For SISO systems we simply find the zeroes as the values s_0 which results in that $H(s_0) = 0$, where $y = H(s)u$ is the transfer function model of the system. This can be extended to MIMO systems as follows:

Definisjon 2.8 (Zeroes and transfer matrix)
s_i is defined as a zero for the transfer matrix $H(s)$ if the rank of $H(s_i)$ is less than the natural (maximal) rank of $H(s)$.

We say that the transfer matrix loses rank if the system is excited a control input with "frequency" equal to the system zero. The effect of this control will then be invisible on at least one of the system outputs.

Notice, that the transfer function $h(s)$ in a SISO system will be equal to zero if the system is exited a control input with such a frequency, i.e., $y(s_i) = h(s_i)u(s_i) = 0$ and $h(s_0) = 0$.

Definisjon 2.9 (Zero polynomial and zeroes from transfer matrix)
The zero polynomial $\rho(s)$ is given as the largest common divisor (numerator) to the under determinants of order r_H for the transfer matrix $H(s)$, where r_H is the natural rank of $H(s)$, assumed that all under determinants are justified such that they have the pole polynomial as denominator.

the natural rank of $H(s)$ is given by the rank of $H(s)$ for all s except for the singularities given by the zeroes s_i. The natural rank of $H(s) \in \mathbb{R}^{m\times r}$ is normally given by

$$r_H = \min(m, r), \tag{2.4}$$

where m is the number of outputs (variables in the vector y) and where r is the number of control inputs (variables in the vector u).

The zero polynomial in factorized form is given by

$$\rho(s) = \prod_{i=1}^{n_n}(s - s_i), \tag{2.5}$$

where $s_i \; \forall \; i = 1, \ldots, n_n$ are the system zeroes. The system zeroes are given as the roots of the zero polynomial.

The transfer matrix model of the system is given by $y(s) = H(s)u(s)$ where $y(s) \in \mathbb{R}^m$ is the system output vector and $u(s) \in \mathbb{R}^r$ is the system input vector (control vector). The normal rank of $H(s)$ is then given by $r_H = \min(m, r)$. The rank of $H(s)$ is less than r_H only for $s = s_i$ equal to the system zeroes.

Merknad 2.1 (Zeroes for non singular (invertible) transfer matrix)
In the case that $H(s)$ is invertible and thereby quadratic, then we can find the zeroes of a minimal realization of $H(s)$ as the poles to $H^{-1}(s)$. I.e., the zeroes for $H(s)$ is in this case found simply found as the roots to the zero polynomial $\rho(s) = \det H(s) = 0$.

Merknad 2.2 (Minimum-phase and non-minimum-phase system)
If the system zeroes are stable, i.e., lies in the left half of the complex plane, then we say that the system is a minimum-phase system. If all or some of the zeroes lies in the right half of the complex plane, the system is said to be a **non-minimum-phase** *system.*

2.5 More about zeroes

1. It is important to notice that the system zeroes are generally not changed by feedback control. This yields both state feedback and output feedback. Example 2.1 illustrates this as well as the effect of zeroes in the right half plane.

2. It is furthermore important to note that the system $\dot{x} = Ax + Bu$ and $y = x$ controlled with state feedback, e.g., $u = G(x^0 - x)$, does not have transmission zeroes, i.e. zeroes from x^0 to the output y. This is one reason for the good robustness properties of Linear Quadratic (LQ) optimal control, i.e., at least $60°$ phase margin and gain margin of $\frac{1}{2}$ ore more. In general,note also that a system with $D = I$ and $E = 0$ does not have zeroes.

3. We usually have zeroes in systems with fewer control inputs (ore outputs) than states, or when $E \neq 0$.

4. Note also that a system may have zeroes at infinity, i.e., $s_0 = \pm\infty$ zeroes. Such zeroes is mostly of interests in root locus analysis, i.e., the investigation of the movement of poles and zeroes in the complex plane by varying the feedback parameters. Zeroes at infinity are not found by the method which is based on the transfer matrix. The method based on the state space model and the generalized eigenvalue problem also finds zeroes at infinity.

Teorem 2.5.1 (Zeroes in open loop and closed loop systems)
Given a system described by

$$\dot{x} = Ax + Bu, \tag{2.6}$$
$$y = Dx + Eu, \tag{2.7}$$

which is controlled by the state feedback

$$u = -Gx + u^0. \tag{2.8}$$

The open loop, uncontrolled system, given by $y = H_p u$ where

$$H_p = D(sI - A)^{-1}B + E, \tag{2.9}$$

have the same zeroes as the feedback controlled closed loop system given by $y = H_{cl} u^0$, where

$$H_{cl} = (D - EG)(sI - (A - BG))^{-1}B + E. \tag{2.10}$$

Proof 2.1 *The closed loop system described with*

$$\dot{x} = (A - BG)x + Bu^0, \tag{2.11}$$
$$y = (D - EG)x + Eu^0. \tag{2.12}$$

This can be written as

$$\overbrace{\begin{bmatrix} sI - (A - BG) & -B \\ D - EG & E \end{bmatrix}}^{S} \begin{bmatrix} x(s) \\ u(s) \end{bmatrix} = \begin{bmatrix} 0 \\ y(s) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{2.13}$$

when s is a zero, i.e., $y(s) = 0$. The zeroes of the controlled system is described by those values s which results in that the matrix S loses rank below the natural rank which is $\min(n+m, n+r)$. *The zeroes are then found by* $\det(S) = 0$.

In order to investigate the relationship between the closed loop system zeroes and the open loop system zeroes, we use that

$$\begin{bmatrix} sI - (A - BG) & -B \\ D - EG & E \end{bmatrix} = \begin{bmatrix} sI - A & -B \\ D & E \end{bmatrix} \begin{bmatrix} I & 0 \\ -G & I \end{bmatrix} \tag{2.14}$$

This means that

$$\det(S) = \det \begin{bmatrix} sI - (A - BG) & -B \\ D - EG & E \end{bmatrix} = \det \begin{bmatrix} sI - A & -B \\ D & E \end{bmatrix} \tag{2.15}$$

We have here used that $\det(AB) = \det(A)$ *for two matrices A and B with suitable dimensions, and if B is non-singular.*

This means that the zeroes of the controlled system is identical to the zeroes of the open loop uncontrolled system, i.e., zeroes does nor change by feedback.

2.6 Examples

Example 2.1 (Effect of feedback: SISO system)
Given a system described by the transfer function

$$h_p(s) = \frac{1-s}{1+s}. \tag{2.16}$$

This system have a loop transmission zero at $s = 1$ and a pole in $s = -1$. We say that the zero is located in the right half plane. The system is therefore a non-minimum phase system and we could have limitations in the feedback gain and the speed response of the control system.

We want to control the system with a proportional, P-controller, i.e.,

$$u = g(y^0 - y), \tag{2.17}$$

where y^0 is the reference and $g = K_p$ is the proportiolnal gain constant. The closed loop system is therefore described by

$$\frac{y}{y^0} = h_{cl}(s) = \frac{h_p(s)h_r(s)}{1-(-1)h_p(s)h_r(s)} = \frac{g(1-s)}{(1-g)s+1+g}, \tag{2.18}$$

where we have used negative feedback.

As we see the closed loop system have a zero at $s = 1$, i.e., unchanged and identical with the zero of the open loop system. This is general, the locations of zeroes are not changed by feedback. The pole of the feedback system is

$$s_{cl} = -\frac{1+g}{1-g}. \tag{2.19}$$

We demand stability of the closed loop system, i.e. we require $s_{cl} < 0$. This is satisfied for

$$-1 < g < 1. \tag{2.20}$$

This implies that the speed of the control system is limited. For this example it implies that we will have problems with an inverse response, because the system is non-minimum phase and that the system have a right hand transmission zero. As we see, the system have an inverse response because the gain at time zero, $t = 0$ is given by

$$h_{cl}(s = \infty) = \frac{-g}{1-g} = \rightarrow -\infty \quad når \quad g \rightarrow 1 \tag{2.21}$$

and that the system have a gain with the opposite sign given by $h_{cl}(s = 0) = \frac{g}{1+g}$. This means that the inverse response increases against infinity for increasing gain g, i.e. when $g \rightarrow 1$. At the same time we obtain faster closed loop time response because the pole of the closed loop system move to the left in the complex plane. i.e. $s_{cl} \rightarrow -\infty$ when $g \rightarrow 1$. The problem is that we cannot obtain fast closed loop response and small inverse response at the same time. This is illustrated in Figure 2.1. We also see from Figure 2.2 that the amount of control increases as $g \rightarrow 1$.

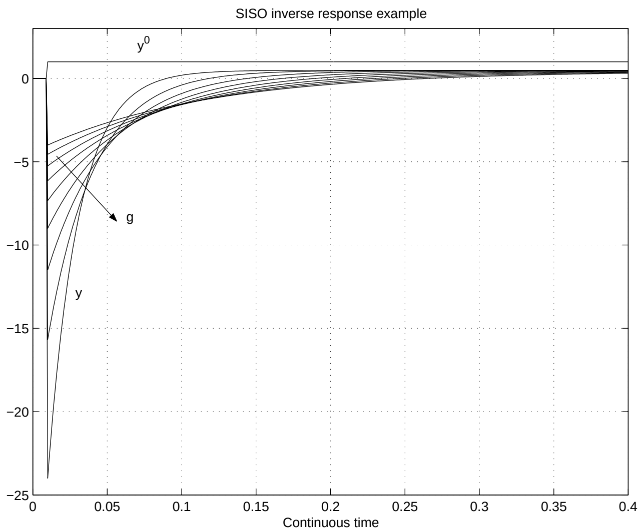

Figure 2.1: Step response simulation of a system with $h_p(s) = \frac{1-s}{1+s}$ and $u = g(y^0 - y)$ for varying proportional coefficient $0.8 < g < 0.96$.

Example 2.2 (PI-control of non-minimum-phase SISO system)
Given a system described by the transfer function

$$h_p(s) = \frac{1-2s}{s^2+3s+2} = \frac{1-2s}{(s+1)(s+2)}, \tag{2.22}$$

which are to be controlled by a PI-controller given by

$$h_c(s) = K_p \frac{1+T_i s}{T_i s}. \tag{2.23}$$

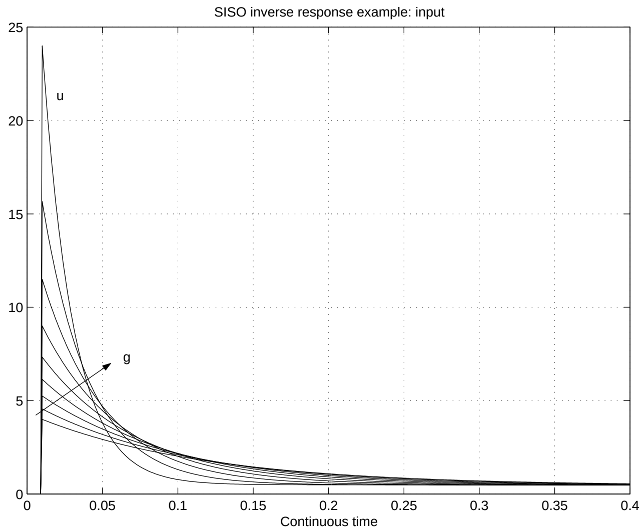

Figure 2.2: The input $u = g(y^0 - y)$ after a step response simulation of a system with $h_p(s) = \frac{1-s}{1+s}$ for varying proportional coefficient $0.8 < g < 0.96$.

Often a common choice is to chose the integral time T_i such that the dominating time constant in the process is canceled, and such that the loop transfer function is, $h_0 = h_p h_c$, at the same time is simplified. The system have two poles/eigenvalues, $s_1 = -1$ and $s_2 = -2$ and therefore also two time constants, e.g., $T_1 = -\frac{1}{s_1} = 1$ and $T_2 = -\frac{1}{s_2} = \frac{1}{2}$. We then have that (with $T_i = T_1 = 1$) that

$$h_0(s) = h_p h_c = \frac{1-2s}{(s+1)(s+2)} K_p \frac{1+T_i s}{T_i s} = \frac{K_p}{T_i} \frac{1-2s}{s(s+2)}, \tag{2.24}$$

where we have chosen $T_i = 1$. Vi kan nå finne krav til proporsjonalkonstanten, K_p, ved å kreve stabilitet av det lukkede systemet, dvs. systemet fra referansen, r, til utgangen, y. Vi har at transferfunksjonen fra r til y i ett reguleringssystem med negativ tilbakekopling er gitt ved

$$\frac{y}{r} = \frac{h_0}{1+h_0} = \frac{\frac{K_p}{T_i}\frac{1-2s}{s(s+2)}}{1 + \frac{K_p}{T_i}\frac{1-2s}{s(s+2)}} = \frac{\frac{K_p}{T_i}(1-2s)}{s^2 + 2(1 - \frac{K_p}{T_i})s + \frac{K_p}{T_i}} \tag{2.25}$$

Det kan vises at ett 2. grads polynom, $s^2 + a_1 s + a_0 = 0$ har røtter i venstre halvplan (stabilt system) dersom koeffisientene er positive, dvs. slik at $a_1 > 0$ og $a_0 > 0$. Dette kan vises ved å studere polynomet, $(s+\lambda_1)(s+\lambda_2) = s^2 + (\lambda_1+\lambda_2)s + \lambda_1\lambda_2 = 0$ som har røtter $s_1 = -\lambda_1$ og $s_2 = -\lambda_2$. Dersom røttene skal ligge i venstre halvplan, dvs. $s_1 < 0$ og $s_2 < 0$ må vi ha at $\lambda_1 > 0$ og $\lambda_2 > 0$. Dette betyr igjen at koeffisientene må være positive, dvs. $a_0 = \lambda_1\lambda_2 > 0$ og $a_1 = \lambda_1 + \lambda_2 > 0$.

Vi får følgende krav til K_p:

$$2(1 - \frac{K_p}{T_i}) > 0 \;\; og \;\; \frac{K_p}{T_i} > 0. \tag{2.26}$$

Dette gir

$$0 < \frac{K_p}{T_i} < 1. \tag{2.27}$$

Vi har nå simulert det lukede reguleringssystemet for forskjellige verdier for K_p *etter at vi påtrykker et enhetssprang i referansen. Resultatet er vist i figur 2.3. Vi ser at systemet får mer oversving og oscillatorisk oppførsel når* K_p *øker mot en. Samtidig ser vi at systemet får en større og større inversrespons som starter ved tiden* $t = 0$*. Inversrespons er et typisk fenomen for systemer med nullpunkt i høyre halvplan.*

Vi ser av figuren at det ikke er enkelt å samtidig få til rask innsvingning, lite oversving og liten inversrespons. Grunnen til disse problemene er at systemet har ett nullpunkt i høyre halvplan. Inversresponsen i prosessen kan vi ikke gjre noe med. Den forefinnes ogs i settpunkts-responsen til det lukkede (regulerte) systemet. Litt prving og feiling med valg av K_p *gir følgende innstilling:*

$$K_p = 0.42, \quad T_i = 1. \tag{2.28}$$

Denne innstillingen gir en forsterkningsmargin,$GM = 2.8$ $[dB]$*, og en fasemargin,* $PM = 71°$*.*

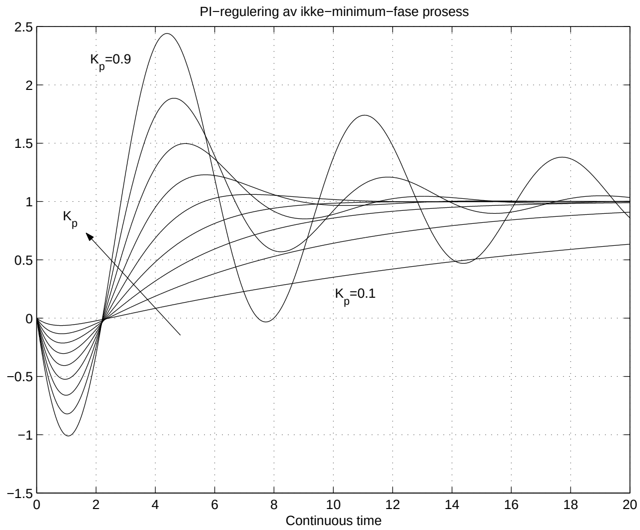

Figure 2.3: Unit step response (in the reference) simulation of a control system with process model, $h_p(s) = \frac{1-2s}{(s+1)(s+2)}$, and PI-controller $h_c(s) = K_p \frac{1+T_i s}{T_i s}$ with $T_i = 1$ and for varying proportional coefficients in the interval, $0.1 \leq K_p \leq 0.9$. Figuren er generert av MATLAB scriptet **siso_zero_ex.m**.

Example 2.3 (PI-regulering av ikke-minimum-fase SISO system)
Gitt et system beskrevet med transferfunksjonen

$$h_p(s) = \frac{1-2s}{s^2+3s+2} = \frac{1-2s}{(s+1)(s+2)}. \tag{2.29}$$

Systemets frekvensrespons er gitt ved

$$h_p(j\omega) = |h_p(j\omega)| e^{j\angle h_p(j\omega)}, \tag{2.30}$$

der fase og amplitude-karakteristikkene er gitt ved

$$\angle h_p(j\omega) = -(\arctan(2\omega) + \arctan(\omega) + \arctan(\frac{\omega}{2})), \tag{2.31}$$

$$|h_p(j\omega)| = \frac{\sqrt{1+4\omega^2}}{\sqrt{1+\omega^2}\sqrt{4+\omega^2}}. \tag{2.32}$$

Fase kryss-frekvensen (kritisk frekvens), ω_{180}, er da gitt ved den frekvens der fasen er -180°, dvs., $\angle h_p(j\omega_{180}) = -\pi$. Den kritiske forsterkning, K_{cu}, er da den forsterkning som er slik at $K_{cu}|h_p(j\omega_{180})| = 1$. Parametrene K_{cu} og ω_{180} kan f.eks. finnes vha. MATLAB funksjonen margin. Vi får

$$\omega_{180} = 1.8708, \tag{2.33}$$
$$K_{cu} = 1.5. \tag{2.34}$$

Vi kan nå enkelt finne parametrene i en PI-regulator gitt ved

$$h_c(s) = K_p \frac{1+T_i s}{T_i s}. \tag{2.35}$$

vha. Ziegler-Nichols metode. Dvs.

$$K_p = \frac{K_{cu}}{2.2} = 0.68, P_u = \frac{2\pi}{\omega_{180}} = 3.36, T_i = \frac{P_u}{1.2} = 2.79. \tag{2.36}$$

Det viser seg ved simulering at responsen i y blir relativt dårlig med dette valg av PI-regulator parametre.

Det lukkede systemet kan videre analyseres som følger. Transferfunksjonen fra r til y er gitt ved:

$$\frac{y}{r} = \frac{h_0}{1+h_0} = \frac{\frac{K_p}{T_i}(1-2s)(1+T_i s)}{s^3 + (3-2K_p)s^2 + (K_p - 2\frac{K_p}{T_i} + 2)s + \frac{K_p}{T_i}}. \tag{2.37}$$

Eksempel 2.6.1 (Styrbarhet av system med to like modi)
Gitt et system $\dot{x} = Ax + Bu$ der

$$A = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}. \tag{2.38}$$

Vi skal vise at et slikt system ikke er styrbart for noen $B = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$.

Vi har at $AB = \lambda B$ og dermed at styrbarhetsmatrisen er gitt ved

$$C_2 = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} B & \lambda B \end{bmatrix}. \tag{2.39}$$

Systemet er ikke styrbart fordi rang(C_2) $< n = 2$. Forsøk å argumentere for dette ved fysiske betraktninger.

Eksempel 2.6.2 (Styrbarhet av system med tre like modi)
Vi skal vise at et system med

$$A = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix} \tag{2.40}$$

ikke er styrbart for noen $B = \begin{bmatrix} b_1 & b_2 & b_3 \end{bmatrix}^T$. *Styrbarhetsmatrisen er i dette tilfellet gitt ved*

$$C_3 = \begin{bmatrix} B & AB & A^2B \end{bmatrix} = \begin{bmatrix} b_1 & \lambda b_1 & \lambda^2 b_1 \\ b_2 & \lambda b_2 + b_3 & \lambda^2 b_2 + 2\lambda b_3 \\ b_3 & \lambda b_3 & \lambda^2 b_3 \end{bmatrix} \tag{2.41}$$

Vi ser at rekke en i C_3 *er lik rekke tre multiplisert med faktoren* $\frac{b_1}{b_3}$. *Vi har dermed at* $rang(C_3) < n = 3$. *Systemet er dermed ikke styrbart.*

Dersom systemet endres til (Jordan form)

$$A = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix} \tag{2.42}$$

kan vi vise at systemet er styrbart for alle $B = \begin{bmatrix} b_1 & b_2 & b_3 \end{bmatrix}^T \neq 0$.

Eksempel 2.6.3 (Inversrespons i tilstandsrom og transferfunksjon)
Gitt et system med tilstandsrommodell

$$\dot{x} = -\frac{1}{T}x + k\frac{T+\tau}{T^2}u, \tag{2.43}$$

$$y = x - \frac{k\tau}{T}u. \tag{2.44}$$

Dette er ekvivalent med transferfunksjonsmodellen

$$\frac{y}{u} = k\frac{1-\tau s}{1+Ts}. \tag{2.45}$$

Dette systemet har en inversrespons på grunn av nullpunktet, $s_0 = \frac{1}{\tau}$ *i høyre halvplan. Merk at inversresponsen* $1-\tau s$ *er en approksimasjon til en transportforsinkelse fordi* $e^{-\tau s} \approx 1-\tau s$. *Modellen (2.45) er ett gunstig utgangspunkt for regulatorsyntese.*

Eksempel 2.6.4 (Inversrespons og modellrediksjon ved halveringsregel)
Gitt et system beskrevet med transferfunksjonen

$$h_p(s) = \frac{1-2s}{(s+1)(s+2)} = k\frac{1-\tau s}{(1+T_1 s)(1+T_2 s)}, \tag{2.46}$$

der

$$k = \frac{1}{2},\ \tau = 2,\ T_1 = 1,\ T_2 = \frac{1}{2}. \tag{2.47}$$

En god approksimasjon for regulatorsyntese er

$$h_p(s) = k\frac{1-\tau s}{1+T_1 s}, \tag{2.48}$$

der $k = \frac{1}{2}$ *og* τ *og* T_1 *finnes fra "halveringsregelen".*

$$\tau := \tau + \frac{1}{2}T_2 = 2 + \frac{1}{4} = \frac{9}{4}, \tag{2.49}$$

$$T_1 := T_1 + \frac{1}{2}T_2 = 1 + \frac{1}{4} = \frac{5}{4}. \tag{2.50}$$

En god PI-regulator innstilling er dermed gitt ved

$$T_i = T_1 = \frac{5}{4} \approx 1.25, \tag{2.51}$$

og

$$K_p = \frac{1}{2}\frac{T_1}{k\tau} = \frac{5}{9} \approx 0.56. \tag{2.52}$$

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Part II

OPTIMAL CONTROL

Chapter 3

Introduction to Continuous Time Linear Quadratic Optimal Control

3.1 Introduction to linear quadratic optimal control

We shall in this section give a presentation of the continuous time Linear Quadratic (LQ) optimal control problem and its solution.

Problem 3.1 (Linear Quadratic Optimal Control)
Assume that the process is modeled by

$$\dot{x} = Ax + Bu, \tag{3.1}$$

with known initial state $x(t = t_0) = x_0$, and that we want a control specified by

$$u = Gx, \tag{3.2}$$

which gives a minimum of the Linear Quadratic (LQ) performance criterion or performance index

$$J = \int_{t_0}^{t_1} (x^T Q x + u^T P u) dt, \tag{3.3}$$

with long or infinite settling time t_1.
$\triangle$

We will in this section for the sake of simplicity putting $t_0 = 0$. Long settling time means that the time interval $[0, t_1 >$ is assumed to be greater than the time constants of the process, or simply infinity.

We will now show that the solution to this problem gives an expression for the feedback matrix G which when applied to the system yields some remarkable properties of the closed loop system.

Substitute the control $u = Gx$ into the performance index. We have

$$\dot{x} = (A + BG)x, \tag{3.4}$$

$$J = \int_0^\infty x^T(Q + G^T PG)xdt. \tag{3.5}$$

The solution of Equation (3.1) is given by

$$x = e^{(A+BG)t}x_0, \tag{3.6}$$

where x_0 is the initial values at time zero, i.e. $x_0 = x(t = 0)$. Substitute the solution into the performance index and we get

$$J = x_0^T[\int_0^\infty e^{(A+BG)^T t}(Q + G^T PG)e^{(A+BG)t}dt]x_0 \overset{\text{def}}{=} x_0^T Rx_0, \tag{3.7}$$

where we have defined

$$R \overset{\text{def}}{=} \int_0^\infty e^{(A+BG)^T t}(Q + G^T PG)e^{(A+BG)t}dt. \tag{3.8}$$

We want a feedback matrix G such that the performance index reach a minimum value. Hence, the performance index J must be finite. This means that the closed loop system matrix $A + BG$ must be stable.

We know from observability analysis that R is the observability Gramian for the system described by the pair $(\sqrt{Q + G^T PG}, A+BG)$ and that this Gramian satisfy the following Lyapunov matrix equation

$$(A + BG)^T R + R(A + BG) + Q + G^T PG = 0. \tag{3.9}$$

Define the following scalar function

$$\begin{aligned}\tilde{J} &= x_0^T[(A + BG)^T R + R(A + BG) + Q + G^T PG]x_0 \\ &= tr(x_0 x_0^T[(A + BG)^T R + R(A + BG) + Q + G^T PG]).\end{aligned} \tag{3.10}$$

The minimization of $\tilde{J}$ with respect to the feedback matrix G is the same as to minimize the performance index J with respect to G. The following can be used to see this

$$R = [(A + BG)^T R + R(A + BG) + Q + G^T PG] + R. \tag{3.11}$$

Premultiplication with x_0^T and postmultiplication with x_0 gives

$$J = \tilde{J} + J, \tag{3.12}$$

where J is defined by (3.7) and $\tilde{J}$ is defined in (3.10). Hence, we have

$$\min_G J = \min_G \tilde{J} + \min_G J, \tag{3.13}$$

which is equivalent to minimize $\tilde{J}$ with respect to G. The minimum of $\tilde{J}$ with respect to G is determined from

$$\frac{d\tilde{J}}{dG} = x_0 x_0^T(2B^T R + 2PG) = 0. \tag{3.14}$$

We have for the minimum that G is given by

$$G = -P^{-1}B^T R, \tag{3.15}$$

and which substituted into the Lyapunov equation gives

$$A^T R + RA - RHR + Q = 0, \tag{3.16}$$

$$H = BP^{-1}B^T, \tag{3.17}$$

which is the famous matrix Algebraic Riccati Equation (ARE). The ARE is named after Count Jacopo Francesco Riccati and his original paper published in 1724. See Bittanti (1989).

3.2 Some simple examples

Example 3.1 (Design of LQ optimal PI controller)
Assume that the process is modeled by

$$\dot{x} = ax + bu, \tag{3.18}$$

$$y = x. \tag{3.19}$$

The problem is to design a LQ optimal PI-controller for the process. A state space formulation of a PI controller is given by

$$\dot{z} = y_0 - y, \tag{3.20}$$

$$u = g_1 x + g_2 z, \tag{3.21}$$

where $g_1 = K_p$ *and* $g_2 = \frac{K_p}{T_i}$.

The first step in the solution procedure is to make an augmented model for the process and the controller. We have

$$\dot{\tilde{x}} = \begin{bmatrix} a & 0 \\ -1 & 0 \end{bmatrix} \tilde{x} + \begin{bmatrix} b \\ 0 \end{bmatrix} u + \begin{bmatrix} 0 \\ 1 \end{bmatrix} y_0, \tag{3.22}$$

where

$$\tilde{x} = \begin{bmatrix} x \\ z \end{bmatrix}. \tag{3.23}$$

The second step in the solution procedure is to choose a Linear Quadratic performance index. We will choose a diagonal weighting matrix Q *for the augmented state vector. We have*

$$J = \int_0^T (\tilde{x}^T \begin{bmatrix} q_{11} & 0 \\ 0 & q_{22} \end{bmatrix} \tilde{x} + u^T pu)dt. \tag{3.24}$$

We will now choose the settling time T *to be large compared to the time constants in the augmented system. The solution to this infinite time horizon LQ problem can then be found by solving the ARE.*

The third step is to solve the Algebraic Riccati Equation (ARE) for the optimal control, $u = G\tilde{x}$, that minimize the quadratic performance index. I.e. we have to solve

$$H = BP^{-1}B^T, \tag{3.25}$$
$$A^T R + RA - RHR + Q = 0, \tag{3.26}$$
$$G = -P^{-1}B^T R. \tag{3.27}$$

We have

$$\begin{bmatrix} a & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} \\ r_{12} & r_{22} \end{bmatrix} + \begin{bmatrix} r_{11} & r_{12} \\ r_{12} & r_{22} \end{bmatrix} \begin{bmatrix} a & 0 \\ -1 & 0 \end{bmatrix} - \tag{3.28}$$
$$\begin{bmatrix} r_{11} & r_{12} \\ r_{12} & r_{22} \end{bmatrix} \begin{bmatrix} h & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} \\ r_{12} & r_{22} \end{bmatrix} + \begin{bmatrix} q_{11} & 0 \\ 0 & q_{22} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}. \tag{3.29}$$

where $h = bp^{-1}b$. We then get

$$2ar_{11} - 2r_{12} - hr_{11}^2 + q_{11} = 0, \tag{3.30}$$
$$ar_{12} - r22 - r_{11}hr_{12} = 0, \tag{3.31}$$
$$q_{22} - hr_{12}^2 = 0. \tag{3.32}$$

The control is given by

$$u = -p^{-1}br_{11}x - p^{-1}br_{12}z. \tag{3.33}$$

This gives

$$r_{12} = \pm\sqrt{\frac{q_{22}}{h}}, \tag{3.34}$$
$$r_{11} = \frac{a \pm \sqrt{a^2 + h(q_{11} - 2r_{12})}}{h}, \tag{3.35}$$
$$r_{22} = ar_{11} - hr_{11}r_{12}. \tag{3.36}$$

We have to chose the positive definite (maximum) solution to the ARE. Hence

$$r_{12} = -\sqrt{\frac{q_{22}}{h}}, \tag{3.37}$$
$$r_{11} = \frac{a + \sqrt{a^2 + h(q_{11} - 2r_{12})}}{h}, \tag{3.38}$$
$$r_{22} = ar_{12} - hr_{11}r_{12}. \tag{3.39}$$

We have

$$g_1 = -\frac{b}{p}r_{11}, \tag{3.40}$$
$$g_2 = -\frac{b}{p}r_{12} = sgn(b)\sqrt{\frac{q_{22}}{p}}. \tag{3.41}$$

Note that the external set-point signal y_0 was put to zero when designing the LQ optimal PI controller. However, the controller can be applied to a plant with $y_0 \neq 0$. However, in this case the solution is not necessary optimal.

Example 3.2 (Double integrator)
Consider an idealized angular position control system where the position of the rotation shaft is controlled by the torque applied, with no friction in the system. The equation of motion is given by

$$J\ddot{\theta} = T, \tag{3.42}$$

where θ is the angular position, T is the applied torque and J is the moment of inertia of the rotating parts. Define

$$x_1 = \theta, \tag{3.43}$$
$$x_2 = \dot{\theta}, \tag{3.44}$$
$$u = T, \tag{3.45}$$
$$b = \frac{1}{J}. \tag{3.46}$$

We have the following state space model

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ b \end{bmatrix} u. \tag{3.47}$$

We choose the following LQ index

$$J = \int_0^\infty (\begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} q_{11} & 0 \\ 0 & q_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + u^T P u) dt, \tag{3.48}$$

which is equivalent to

$$J = \int_0^\infty (q_{11} x_1^2 + q_{22} x_2^2 + p u^2) dt. \tag{3.49}$$

The ARE is in this case given by

$$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} \\ r_{12} & r_{22} \end{bmatrix} + \begin{bmatrix} r_{11} & r_{12} \\ r_{12} & r_{22} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - \tag{3.50}$$

$$\begin{bmatrix} r_{11} & r_{12} \\ r_{12} & r_{22} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & h \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} \\ r_{12} & r_{22} \end{bmatrix} + \begin{bmatrix} q_{11} & 0 \\ 0 & q_{22} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}. \tag{3.51}$$

where $h = bp^{-1}b$. We get

$$\begin{bmatrix} -hr_{12}^2 + q_{11} & r_{11} - hr_{22}r_{12} \\ r_{11} - hr_{22}r_{12} & 2r_{12} - hr_{22}^2 + q_{22} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}. \tag{3.52}$$

For this 2nd order example, there are $2n = 4$ solutions to the ARE. We want the unique positive definite solution, corresponding to the stable closed loop eigenvalues. Hence

$$r_{12} = \sqrt{\frac{q_{11}}{h}}, \tag{3.53}$$

$$r_{22} = \sqrt{\frac{2r_{12} + q_{22}}{h}} = \sqrt{\frac{2\sqrt{\frac{q_{11}}{h}} + q_{22}}{h}}. \tag{3.54}$$

The optimal control is given by

$$u = -P^{-1}B^{T}Rx = g_1x_1 + g_2x_2, \tag{3.55}$$

where the feedback gain values g_1 and g_2 are given by

$$g_1 = -\tfrac{b}{p}r_{12} = -\tfrac{b}{\sqrt{b^2}}\sqrt{\tfrac{q_{11}}{p}}, \tag{3.56}$$

$$g_2 = -\tfrac{b}{p}r_{22} = -\tfrac{b}{b^2}\sqrt{\tfrac{2r_{12}+q_{22}}{p}}. \tag{3.57}$$

Note that the fractions $\frac{q_{11}}{p}$ and $\frac{q_{22}}{p}$ are involved in the feedback elements.

Chapter 4

Optimal Control of Continuous Time Systems

4.1 The maximum principle for continuous time systems

Given a process

$$\dot{x} = f(x, u, t). \tag{4.1}$$

We will assume that the initial state is given, i.e., the initial value of the state vector $x(t_0)$ is given (known).

For the final state vector $x(t_1)$ we consider the following cases

1. $x(t_1)$ given.
2. $x(t_1)$ should belong to a specified domain.
3. $x(t_1)$ is completely free.
4. $x(t_1)$ can be weighted in an optimal criterion.

The optimal criterion os of the form

$$J = S(x(t_1)) + \int_{t_0}^{t_1} L(x, u, t)dt, \tag{4.2}$$

where we assume that the starting time t_0 is given. Often we only consider $t_0 = 0$. The final time instant t_1 can be given ore a free variable.

The optimal control problem is now to minimize (alternatively maximize) the optimal control criterion J with respect to the control function $u(t)$ over the time horizon $t_0 \leq t \leq t_1$. This can be formulated as follows

$$\min_{u \in U} J \tag{4.3}$$

where U denotes the control space. Note that we have the process model $\dot{x} = f(x, u, t)$ as a bi-constraint to the optimization problem.

The first which is defined is the so called Hamiltonian function

$$H(x, p, u, t) = L(x, u, t) + p^T f(x, u, t), \tag{4.4}$$

where we have included and defined the so called impulse vector $p(t)$. The impulse vector can be viewed as an Lagrange multiplier which is used in order to reformulate the optimization problem with constraints to a problem without constraints. The optimal control function, $u(t)$, may now be found as the optimum of the hamiltonian function (4.4). This will be shown in the following.

In order for the control function $u(t) \in U$ to be the optimal control which minimizes J it is necessary that:

- $$\dot{x} = \frac{\partial H}{\partial p}, \tag{4.5}$$

with given initial state $x(t_0)$. The final state condition $x(t_1)$ may be as specified above.

The impulse vector satisfies

$$\dot{p} = -\frac{\partial H}{\partial x}, \tag{4.6}$$

The border conditions for the impulse vector is only given and defined at the final time t_1. We consider the following case

$$p(t_1) = \left. \tfrac{\partial S}{\partial x} \right|_{t_1} \tag{4.7}$$

- The Hamiltonian function H must have a global minimum with respect to the control function $u(t) \in U \ \forall \ t_0 \leq t \leq t_1$ such that

$$u^* = \arg \min_{u(t) \in U} H(x, p, u, t), \tag{4.8}$$

  is the optimal control function.

- Conditions for minimum is then

$$\frac{\partial H}{\partial u} = 0, \tag{4.9}$$

  and in order for a minimum problem

$$\frac{\partial^2 H}{\partial u^2} > 0. \tag{4.10}$$

- In case that the final time t_1 is not specified, then we must have that

$$H(t_1) = \left. -\tfrac{\partial S}{\partial t} \right|_{t_1} \tag{4.11}$$

  Usually we have that the function $S(x(t_1))$ is independent of time t. In this case this condition simply reduces to

$$H(t_1) = 0 \tag{4.12}$$

The Maximum Principle was first presented by Pontryagin (1956). We will later on use the Maximum Principle in order to solve many linear optimal control problems. The Maximum Principle can also be used to solve non-linear optimal control problems. Note that the optimal solution in (4.8) usually gives an open loop control strategy in which the controls are computed in advance. However, there is important special cases which gives a optimal feedback control structure.

4.2 Linear systems with Quadratic criterions

Given a linear continuous time system described by the state space model

$$\dot{x} = Ax + Bu, \tag{4.13}$$

with initial state vector $x(t_0)$ at initial time instant t_0. The optimal criterion or performance index, valid over the time horizon $t_0 \leq t \leq t_1$, is assumed given by the following Linear Quadratic (LQ) function

$$J = \frac{1}{2}x^T(t_1)Sx(t_1)) + \frac{1}{2}\int_{t_0}^{t_1}[x^TQx + u^TPu]dt. \tag{4.14}$$

This criterion is referred to as a Linear Quadratic (LQ) criterion.

There are some demands for the weighting matrices S, Q and P in order for the optimal problem to have a solution.

First of all, S, Q and P are symmetric weighting matrices. We will also show that the control weighting matrix P must be positive definite. Furthermore, it is sufficient that the weighting matrices S and Q are positive semi definite. We will discuss those demands later.

We start by defining the Hamiltonian function

$$H = \frac{1}{2}(x^TQx + u^TPu) + p^T(Ax + Bu). \tag{4.15}$$

We are to minimize H with respect to u. A necessary condition for a minimum is found by putting the gradient of H with respect to u equal to zero, i.e.,

$$\frac{\partial H}{\partial u} = Pu + B^Tp = 0. \tag{4.16}$$

This gives the following control

$$u = -P^{-1}B^Tp. \tag{4.17}$$

We now have to find an expression for the impulse vector p and we will later on show that there is a relationship $p = Rx$ where R is the solution to a matrix Riccati equation. However, let us first look at the second derivative of H with respect to u, i.e. the sufficient condition for a minimum. We have

$$\frac{\partial^2 H}{\partial u^2} = P. \tag{4.18}$$

The second order derivative is is connection with optimization theory often referred to as the hessian matrix. A condition for that the control given by (4.17) at least should result in a minimum criterion value is that the Hessian matrix is positive definite. This means that we must demand the control weighting matrix to be positive definite, i.e., $P > 0$ in order to guarantee a minimum.

As we see, in order to compute the optimal control from (4.17) we must find en expression for the impulse vector p. The impulse vector p is defined from (4.17). We have

$$\dot{p} = -\frac{\partial H}{\partial x} = -Qx - A^Tp, \tag{4.19}$$

and from equation (4.7) we obtain

$$p(t_1) = \frac{\partial}{\partial x}(\tfrac{1}{2}x^T(t_1)Sx(t_1))\Big|_{t_1} = Sx(t_1). \tag{4.20}$$

As we see, there is a linear relationship between the impulse vector p and the state vector x at the final time instant t_1. We will later on show that this also is the case for all time instants $t_0 \leq t \leq t_1$.

From equation (4.5) we obtain

$$\dot{x} = \frac{\partial H}{\partial p} = Ax + Bu, \tag{4.21}$$

which is identical to the system model, i.e., this gives no further information. We are now putting the optimal control given by (4.17) into(4.21) and obtain

$$\dot{x} = Ax - Hp, \tag{4.22}$$

where we have defined the matrix

$$H = BP^{-1}B^T. \tag{4.23}$$

We will now prove a linear relationship $p = Rx$ between the state vector x and the impulse vector p. By viewing the equations for $\dot{x}$ and $\dot{p}$, we see that they form an autonomous system, i.e.,

$$\begin{bmatrix} \dot{x} \\ \dot{p} \end{bmatrix} = F \begin{bmatrix} x \\ p \end{bmatrix}, \tag{4.24}$$

where the matrix F is given by

$$F = \begin{bmatrix} A & -H \\ -Q & -A^T \end{bmatrix}. \tag{4.25}$$

The matrix F is denoted the Hamiltonian matrix for the autonomous system. This matrix is also very central in connection with the LQ optimal control solution.

We will now show that there is a linear relationship between p and x for all $t_0 \leq t \leq t_1$. This relationship will result in a very simple formulation of the optimal control given by (4.17).

For given border conditions $x(t_1)$ and $p(t_1)$ at the final time $t = t_1$ we have that the solution of the autonomous system is given by

$$\begin{bmatrix} x(t_1) \\ p(t_1) \end{bmatrix} = \Phi(t_1, t) \begin{bmatrix} x \\ p \end{bmatrix}, \tag{4.26}$$

where Φ is the transition matrix

$$\Phi(t_1, t) = e^{F(t_1 - t)} = \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{bmatrix}. \tag{4.27}$$

Hence, we have the following two equations

$$x(t_1) = \Phi_{11}x + \Phi_{12}p, \tag{4.28}$$

$$p(t_1) = \Phi_{21}x + \Phi_{22}p = Sx(t_1), \tag{4.29}$$

where we in (4.29) have used the expression for $p(t_1)$ given by (4.20). Combining Equations (4.28) and (4.29) gives (i.e. we have put $x(t_1)$ given by (4.28) into Equation (4.29)), i.e.

$$p = Rx, \tag{4.30}$$

where

$$R = [\Phi_{22} - S\Phi_{12}]^{-1}[S\Phi_{11} - \Phi_{21}]. \tag{4.31}$$

If we are letting $t = t_1$ in (4.27) then we have that $\Phi(t_1, t_1) = I_{2n}$. This means that the corresponding sub matrices are $\Phi_{11} = \Phi_{22} = I_n$ and $\Phi_{12} = \Phi_{21} = 0_n$. Putting this into (4.31) gives the following border condition for R at the final time instant $t = t_1$, i.e.,

$$R(t_1) = S. \tag{4.32}$$

We have now found that the optimal control is given by

$$u = G(t)x, \tag{4.33}$$

where

$$G(t) = -P^{-1}B^T R(t), \tag{4.34}$$

is the LQ optimal feedback matrix. In order to compute $G(t)$ we have to compute an expression for R. The matrix R is in general time dependent and given by Equation (4.31) with border condition as in Equation (4.32). The matrix R can in principle be computed as in Equation (4.31). However, we will below show that thee exist a method which does not involve the explicit problem of evaluating the transition matrix, i.e. the matrix exponent in (4.31),

We will now show that R satisfies a matrix differential equation, i.e., the so called Riccati equation. From (4.30) we have that

$$\dot{p} = \dot{R}x + R\dot{x}. \tag{4.35}$$

from (4.19), (4.21), (4.30) and (4.33) we find that

$$\dot{x} = (A - BP^{-1}B^T R)x, \tag{4.36}$$

$$\dot{p} = (-Q - A^T R)x. \tag{4.37}$$

Putting $\dot{p}$ and $\dot{x}$ given by Equations (4.36) and (4.37) into equation (4.35) gives,

$$(\dot{R} + A^T R + RA - RBP^{-1}B^T R + Q)x = 0. \tag{4.38}$$

Since the state vector x may be arbitrarily different from zero (i.e. $x \neq 0$), at least close to the initial time $t = t_0$, then we have that

$$A^T R + RA - RBP^{-1}B^T R + Q = -\dot{R}. \tag{4.39}$$

This is a so called matrix differential Riccati equation with border condition as given by Equation (4.32). We see that the matrix R is a solution to the Riccati equation (4.39).

The solution of the Riccati equation is of central importance for the optimal feedback given by (4.33) and (4.34). Hence, it is of importance to note the following moments with the Riccati equation. The Riccati equation have border conditions at the final time, i.e., $R(t_1) = S$. The Riccati equation is therefore solved backward in time, i.e. from the final time t_1 and backward to the present time instant t in order to compute $R(t)$ which is used to compute the present optimal control $u(t) = -P^{-1}B^T R(t)x(t)$. The Riccati equation have $2n$ solutions. From all those $2n$ solutions there is only one unique positive definite and symmetric solution $R > 0$. This positive definite solution R is to be used in order to compute the optimal control.

Furthermore, it can be shown that the minimum value of the optimal criterion over the optimization horizon $t_0 \leq t \leq t_1$ is given by

$$J^* = x(t_0)^T R(t_0)x(t_0). \tag{4.40}$$

As we see, the minimum criterion value is dependent of the initial state $x(t_0)$ as well as the solution of the Riccati equation $R(t_0)$ at time $t = t_0$.

4.3 Constant running time horizon (Receding horizon)

We have in the above Section 4.2 considered a fixed optimization interval $t_0 \leq t \leq t_1$. A special case of great interest is to consider a running constant optimization horizon in which $t_0 = t$ and $t_1 = t + T$ where T is the usually constant prediction horizon. The standard optimization criterion will in this case be of the form

$$J(t) = \frac{1}{2}x(t+T)^T Sx(t+T) + \frac{1}{2}\int_t^{t+T} [x^T Qx + u^T Pu]dt. \tag{4.41}$$

where $S \geq 0$, $Q \geq 0$ and $P > 0$ are symmetric weighting matrices. The weighting matrices may in general be time varying matrices.

From Equations (4.31) and (4.27) we see that R is a function of the time horizon $t_1 - t$. In this case we have that $t_1 - t = T$ is constant and therefore we have that $R = R(T)$ is a constant matrix and not dependent of time t. Furthermore this gives a constant feedback matrix $G = G(T) = -P^{-1}B^T R(T)$. This means that the feedback matrix only is dependent of the constant horizon T, which usually is referred to as the *prediction horizon* in Model Predictive Control (MPC).

Minimization of this criterion with respect to the process model $\dot{x} = Ax + Bu$ with respect to the control vector u gives the optimal control u^* at the present time t, i.e., $u^*(t) = Gx(t)$. However, all the optimal control over the optimization horizon $[t, t+T>$ are computed. However, the optimization problem is recalculated at each new time instant. It can therefore be natural to only use the control u^* at time t. The most important motivation behind this is that the optimal control is simply $u^*(t) = Gx(t)$ where $G = G(T)$ which can be computed off line and in advance.

Basically we have an optimization problem at each time instant t. At the present time t a prediction T time units into the future is performed. Note however, that we does not have any constraints on the inputs ore process outputs ore states, we have that the optimal control is given by $u^* = G(T)x$ where $G(T)$ is constant and only dependent of the prediction horizon, as well as the matrices A, B, P, Q, S. We therefore, in this unconstrained LQ optimization problem, with receding horizon, does not need to recompute the optimal solution. The above discussion also holds for unconstrained Model Predictive Control (MPC).

The optimization control problem with constant running optimization time horizon is referred to as *receding horizon control*.

The above details is described in Balchen (1970). The basic Model Predictive Control (MPC) theory is therefore not new and described in many text books on optimal control theory.

4.4 LQ optimal control with infinite time horizon

A special case of great importance is obtained by putting the horizons to be large, ore infinite, i.e. $T \to \infty$ or $t_1 \to \infty$. This means in practice that the optimization time interval is sufficiently larger than the time constants in the system 8closed loop system), i.e. that t_1 is large. The Riccati equation is a stable matrix differential equation which converges to a constant solution R if the final time t_1 is large. This again means that we obtain a constant feedback matrix G and feedback $u = Gx$. It gives in this case no meaning of weighting the states in infinity, that at time $t_1 = \infty$. We therefore let $S = 0$. It can also be proved that the solution to this problem is independent of S. The optimal criterion becomes in this case

$$J = \frac{1}{2}\int_{t_0}^{\infty}[x^T Q x + u^T P u]dt. \tag{4.42}$$

In this case we say that R is a solution to The Algebraic Riccati Equation (ARE), i.e.,

$$A^T R + RA - RBP^{-1}B^T R + Q = 0, \tag{4.43}$$

because $\dot{R} = 0$ when $t_1 \to \infty$.

If a minimum of the objective J given by Eq. (4.42) exist, then J have to converge against a finite value when time approach infinity. This implies that the state, $x = 0$, when $t \to \infty$ and that the control approaches $u = 0$ because $u = Gx$.

If the system is unstable then $x \to \infty$ when $t \to \infty$. In such a case there will not be a finite value on the objective J and there will not exist an optimal solution.

There exist some requirements to the weighting matrices Q and P for the solution to the LQ optimal control problem to give a stable closed loop (controlled) problem. We have the following important theorem about stability in LQ optimal control systems with infinite setling time (infinite horizon).

Theorem 4.4.1 (Stability of LQ optimal systems)
Given a continuous time invariant system $\dot{x} = Ax + Bu$ and a Linear Quadratic (LQ) objective with infinite time horizon ($t_1 \to \infty$) with weighting matrices $Q = D^T D$ and $P > 0$.

The optimal control is of state feedback type $u^* = Gx$ with feedback gain matrix $G = -P^{-1}B^T R$, where R is the unique positive definite solution to the Algebraic Riccati Equation (ARE). The optimal controller gives a stable closed loop system, i.e. the eigenvalues of $A + BG$ is located in the left part of the complex plane, if and only if the pair (A, B) is stabilizable and the pair (A, D) is detectable.
$\triangle$

Note that in connection with this theorem, that the product $D^T D$ may be a square root factorization of Q. Some times we also equivalently says that the pair $(A, \sqrt{Q})$ should be detectable.

4.5 Solution of the Algebraic Riccati Equation

There exist many methods for solving the Algebraic Riccati Equation (ARE), i.e.,

$$A^T R + RA - R\overbrace{BP^{-1}B^T}^{H} R + Q = 0, \tag{4.44}$$

where $H = BP^{-1}B^T$ is the Hamiltonian matrix.

Possibly the best and most stable method is based on a Schur decomposition of the Hamiltonian matrix. It can be shown that the positive definite solution R of the ARE may be expressed in terms of the eigenvectors connected to the stable eigenvalues of the Hamiltonian matrix F. Furthermore, it is also possibile to find all $2n$ solutions of the ARE from this method, but remember that we usually only need the unique positive definite solution R for control purposes.

Given a real Schur decomposition of the Hamiltonian matrix F, i.e.,

$$\overbrace{\begin{bmatrix} A & -H \\ -Q & -A^T \end{bmatrix}}^{F} \overbrace{\begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix}}^{U} = \overbrace{\begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix}}^{U} \overbrace{\begin{bmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{bmatrix}}^{T}. \tag{4.45}$$

where U and T are real matrices obtained from the real Schur decomposition $F = UTU^T$. The matrix U contains the Schur eigenvectors to the matrix F. Furthermore U is an orthogonal matrix such that $U^{-1} = U^T$. T is an upper block triangular matrix with 1×1 or 2×2 blocks on the diagonal. Real eigenvalues of F is contained in the 1×1 on the diagonal. Complex conjugate eigenvalues of F are contained in 2×2 on the diagonal of T.

Furthermore the Schur decomposition (eigenvalues and eigenvectors) may be ordered such that the eigenvalues (of F and T) may be ordered in an arbitrarily specified order along the diagonal of T. Hence, we may order the Schur decomposition such that all n stable eigenvalues is located in the T_{11} part and all n unstable eigenvelues in T_{22}. We then have that

$$F = UTU^T \tag{4.46}$$

It can be shown that the unique solution to the Riccati equation may be expressed in terms of the Schur eigenvectors of U of the Hamiltonian matrix F. When U_{11} is non-singular, then

$$R = U_{21}U_{11}^{-1}, \tag{4.47}$$

is the unique positive definite solution to the Algebraic Riccati Equation (ARE)

$$A^T R + RA - RHR + Q = 0. \tag{4.48}$$

where $H = BP^{-1}B^T$. Here we assume that the stable eigenvalues of F are located in T_{11} and the Schur decomposition as in (4.45).

It can be shown that the eigenvalues of the closed loop system $A + BG$ where $G = -P^{-1}B^T R$ is given by the eigenvalues of T_{11}, and located on the diagonal (1 × 1 and 2 × 2 blocks on the diagonal of T_{11}.

In connection with the optimal solution to the LQ control problem we want the unique positive definite solution R of the ARE which results in a stable closed loop system. Hence, the Schur decomposition have to be ordered such that the matrix T_{11} contains the stable eigenvalues of the Hamiltonian matrix.

4.6 Linear system with disturbance

Assume that the process can be described by the following linear continuous time state space model

$$\dot{x} = Ax + Bu + Cv, \tag{4.49}$$

$$y = Dx, \tag{4.50}$$

where v is a vector of process noise (disturbances). We will in this section assume that the process noise,v, is colored. This means that v has a mean value different from zero and that the noise is time varying. We will assume that v is measured or estimated in an estimator.

In case when v is withe Gausian noise with zero mean, then it can be shown that the solution to the LQ optimal control problem is identical to the LQ optimal solution which is obtained for $v = 0$. This solution consists as we have shown of a state feedback, $u = G(t)x$.

We will in the following sections show that in case when v is colored then the LQ optimal solution will consists of a feedback from the state, x, and a feed forward from the disturbance,v.

4.6.1 Solution by reformulating the problem as a standard LQ problem

The solution which is described in this section is dependent on a model for the process disturbance. The disturbance may often be modelled. Assume that the surrounding which generates the disturbance may be modelled by a linear state space model of the form

$$\dot{x}_2 = Ex_2 + Fn, \tag{4.51}$$

$$v = Hx_2, \tag{4.52}$$

where n is a rudimentary disturbance which excites the surrounding disturbance model. By rudimentary we mean a stylized noise, e.g. n may be white Gausian noise with zero mean, ore an impulse at time $t_0 = 0$.

In case when the process disturbance which influence upon the process is constant ore slowly varying, then we can describe the noise model simply as

$$\dot{v} = Fn. \tag{4.53}$$

where n is a rudimentary noise process. In case of a constant disturbance the noise model is given by

$$\dot{v} = 0. \tag{4.54}$$

The process model and the disturbance model can be augmented together to a linear state space model given as follows,

$$\dot{\tilde{x}} = \tilde{A}\tilde{x} + \tilde{B}u + \tilde{C}n, \tag{4.55}$$

$$\tag{4.56}$$

where

$$\tilde{A} = \begin{bmatrix} A & CH \\ 0 & E \end{bmatrix}, \tag{4.57}$$

$$\tilde{B} = \begin{bmatrix} B \\ 0 \end{bmatrix}, \tag{4.58}$$

$$\tilde{C} = \begin{bmatrix} 0 \\ F \end{bmatrix}. \tag{4.59}$$

Since the noise vector n is rudimentary it will not influence upon the LQ optimal control problem.

We are now choosing a standard LQ optimal criterion given as follows

$$J = \frac{1}{2}\tilde{x}^T(t_1)\tilde{S}\tilde{x} + \frac{1}{2}\int_{t_0}^{t_1} (\tilde{x}^T\tilde{Q}\tilde{x} + u^T Pu)dt \tag{4.60}$$

where

$$\tilde{Q} = \begin{bmatrix} Q & 0 \\ 0 & Q_2 \end{bmatrix}, \quad \tilde{S} = \begin{bmatrix} S & 0 \\ 0 & S_2 \end{bmatrix}. \tag{4.61}$$

Q and S are weighting matrices for the process state x. Q_2 and S_2 is weighting matrices for the state vector x_2 in the surrounding noise model which generates the disturbance v. We often have little knowledge of how to weight the states in the noise model so often we are putting $Q_2 = 0$ and $S_2 = 0$. In this case the criterion is simply

$$J = \frac{1}{2}x^T(t_1)Sx(t_1) + \frac{1}{2}\int_{t_0}^{t_1} (x^T Qx + u^T Pu)dt \tag{4.62}$$

The solution to the LQ optimal control problem is now given by

$$u = -P^{-1}\tilde{B}^T R\tilde{x}, \tag{4.63}$$

where R is the positive definite solution to the Riccati equation

$$\tilde{A}^T R + R\tilde{A} - R\tilde{B}P^{-1}\tilde{B}^T R + \tilde{Q} = -\dot{R}. \tag{4.64}$$

The boundary conditions for the Riccati equations becomes in this case $R(t_1) = S$. If we does not have any weighting of the final state in the LQ criterion then we have that $S = 0$ and the boundary conditions become $R(t_1) = 0$. This is always reasonable when t_1 is large.

Let us study the LQ optimal solution. The solution to the Riccati equation can be presented as follows

$$R = \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix}, \tag{4.65}$$

where $R_{21} = R_{12}^T$, $R_{11} = R_{11}^T$ and $R_{22} = R_{22}^T$ because R is a symmetric matrix. The optimal control can therefore be written as follows

$$\begin{aligned} u &= -P^{-1}\begin{bmatrix} B^T & 0 \end{bmatrix}\begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix}\begin{bmatrix} x \\ x_2 \end{bmatrix} \\ &= G_1 x + G_2 x_2, \end{aligned} \tag{4.66}$$

where

$$G_1 = -P^{-1}B^T R_{11}, \tag{4.67}$$

$$G_2 = -P^{-1}B^T R_{12}. \tag{4.68}$$

As we see, the LQ optimal control u consists of a feed back from the process state vector x and a feed forward from the state vector x_2 in the surrounding noise model. The solution demands that both state vectors x and x_2 is available, measured ore estimated in state estimators.

By studying the Riccati equation we find that

$$A^T R_{11} + R_{11}A - R_{11}BP^{-1}B^T R_{11} + Q = -\dot{R}_{11}, \tag{4.69}$$

and

$$R_{11}CH + R_{12}E + (A - BP^{-1}B^T R_{11})^T R_{12} = -\dot{R}_{12}. \tag{4.70}$$

We have here used that $Q_2 = 0$ and that R_{11} is symmetric. The boundary conditions becomes $R_{11}(t_1) = S$ and $R_{12}(t_1) = 0$.

We note that the feedback matrix G_1 from the process state vector x is independent of the surrounding noise model which generates the disturbance v. However, on the other side we see that the feed forward from the state x_2 in the noise model is dependent on R_{11} and thereby the feedback.

Assume an infinite time horizon and the noise model $\dot{v} = 0$. Then we have that

$$R_{12} = -(A + BG_1)^{-T} R_{11} C \tag{4.71}$$

and the feed forward matrix from v to u is given by

$$G_2 = P^{-1} B^T (A + BG_1)^{-T} R_{11} C. \tag{4.72}$$

As we see, the optimal solution is only dependent on the solution R_{11} of the stationary algebraic Riccati equation.

4.6.2 Solution by the use of the maximum principle

One advantage of the solution presented in this section is that we does not need an explicit model for the disturbance, v. However, as we will see, the optimal solution is based on known future disturbances. But in case of large or infinite optimization horizon the solution is considerably simplified and consists of feedback from the states, x, and feed forward from measured or estimated disturbances, v.

When using the maximum principle we first define the Hamiltonian function

$$H = \frac{1}{2}(x^T Q x + u^T P u) + p^T (Ax + Bu + Cv). \tag{4.73}$$

We will now assume that the impulse vector p is given by the relationship

$$p = Rx + h, \tag{4.74}$$

where h at this stage is an unknown time varying vector function. Think of the term h as a feed forward function due to the non-zero process disturbances v. Hence, we have

$$\dot{p} = \dot{R}x + R\dot{x} + \dot{h}. \tag{4.75}$$

We now put the equations for $\dot{p}$ and $\dot{x}$ as well as the optimal control $u = -P^{-1}B^T p$ into Equation (4.75)-

From the maximum principle we have that

$$\dot{p} = -\frac{\partial H}{\partial x} = -Qx - A^T p. \tag{4.76}$$

Putting p given by (4.74) ito (4.76) gives

$$\dot{p} = -Qx - A^T \overbrace{(Rx + h)}^{p} = -Qx - A^T R x - A^T h. \tag{4.77}$$

Furthermore, the optimal control is of the form

$$\frac{\partial H}{\partial u} = 0 \;\Rightarrow\; u^* = -P^{-1} B^T p. \tag{4.78}$$

Inserting (4.74) into (4.78) gives

$$u^* = -P^{-1} B^T R x - P^{-1} B^T h. \tag{4.79}$$

As we see, the optimal control is generated through a feedback from the state vector, x, and a feed forward from the feed forward signal vector h. In order to obtain a complete solution we have to find the matrix R and the vector h.

Putting the optimal control given by (4.78) into the process model $\dot{x} = Ax+Bu+Cv$ gives,

$$\dot{x} = (A - BP^{-1}B^T R)x - BP^{-1}B^T h + Cv. \tag{4.80}$$

Inserting (4.80) into the equation for $\dot{p}$ given by (4.75) gives

$$\begin{aligned} \dot{p} = (&\dot{R} + RA + A^T R - RBP^{-1}B^T R)x \\ &+\dot{h} + RCv - RBP^{-1}B^T h. \end{aligned} \tag{4.81}$$

Inserting $\dot{p}$ given by (4.77) gives

$$\begin{aligned} &(\dot{R} + RA + A^T R - RBP^{-1}B^T R + Q)x \\ &+\dot{h} + (A - BP^{-1}B^T R)^T h + RCv = 0. \end{aligned} \tag{4.82}$$

This must be valid for al $x \neq 0$. We also recognize the Riccati equation. Hence, the matrix R is the solution to the Riccati equation and the feed forward signal h is given by a differential equation. We have

$$-\dot{R} = RA + A^T R - RBP^{-1}B^T R + Q, \tag{4.83}$$

$$-\dot{h} = (A + BG_1)^T h + RCv, \tag{4.84}$$

where

$$G_1 = -P^{-1}B^T R. \tag{4.85}$$

The boundary conditions for the differential equations are found as follows. From the maximum principle, Equation (4.7) we find that

$$p(t_1) = \frac{\partial}{\partial x}\left[\, \tfrac{1}{2}x(t_1)^T S x(t_1)\,\right]_{t_1} = Sx(t_1). \tag{4.86}$$

Putting $t = t_1$ into (4.74) gives

$$p(t_1) = R(t_1)x(t_1) + h(t_1). \tag{4.87}$$

This must apply to any end state $x(t_1)$. Ie. by comparing equations (4.86) and (4.87) we find the boundary conditions

$$R(t_1) = S, \tag{4.88}$$

$$h(t_1) = 0. \tag{4.89}$$

Note that we get the special case $R(t_1) = 0$ and $h(t_1) = 0$ if we do not weight the state x at the end time, i.e. set $S = 0$ in the optimal criterion.

We see that equation (4.83) is identical to the Riccati equation which we would find if we did not have a process disturbance v. We see that the process disturbance does not affect the optimum the feedback. This is then also expected because forward shifts does not affect the stability of the system. The stability of a linear system can only be operated by feedback.

The optimal forward link given by equation (ref eq11lm) is however, depending on the solution of the Riccati equation R (ie. depending on the optimal feedback system). Note that the solution of (4.84) is given by

$$h(t_1) = e^{-(A+BG_1)^T(t_1-t)}h(t) - \int_t^{t_1} e^{-(A+BG_1)^T(t_1-\tau)}RCv d\tau. \tag{4.90}$$

We have used equation here (1.9). Equation (4.90) can be solved with respect to $h(t)$. This gives

$$h(t) = (e^{-(A+BG_1)^T(t_1-t)})^{-1}h(t_1) + (e^{-(A+BG_1)^T(t_1-t)})^{-1}\int_t^{t_1} e^{-(A+BG_1)^T(t_1-\tau)}RCv d\tau.$$

We use the identity $(e^A)^{-1} = e^{-A}$ to invert a matrix exponent and we have

$$h(t) = e^{(A+BG_1)^T(t_1-t)}h(t_1) + e^{(A+BG_1)^T(t_1-t)}\int_t^{t_1} e^{-(A+BG_1)^T(t_1-\tau)}RCv d\tau. \tag{4.91}$$

We have the boundary condition (4.89) and thus

$$h(t) = e^{(A+BG_1)^T(t_1-t)}\int_t^{t_1} e^{-(A+BG_1)^T(t_1-\tau)}RCv d\tau. \tag{4.92}$$

This can be simplified to

$$h(t) = \int_t^{t_1} e^{(A+BG_1)^T(\tau-t)}RCv d\tau. \tag{4.93}$$

In order to solve this integral and thus find $h(t)$ we have to know the future disturbances $v(t)$ over the time interval $[t, t_1 >$.

Pay special attention to the steady state solution. By setting $\dot{h} = 0$ into (4.84) we find that

$$h = -(A + BG_1)^{-T}RCv \tag{4.94}$$

We find this answer, and by integrating (4.93) analytical as $t_1 \to \infty$.

This provides a constant forward link from the disturbance/interference v. It can further be shown that this is the solution of the integral (4.93) if v is constant over the time interval $[t, t_1 >$ and if we let $t_i \to \infty$. We then have the optimal control

$$u = G_1x + G_2v \tag{4.95}$$

where

$$G_1 = P^{-1}B^T(A + BG_1)^{-T}RC. \tag{4.96}$$

4.7 Optimal tracking systems

We will in this section study optimal tracing systems. With tracking systems we mean that the output y of the system is to follow a prescribed reference r in such a way that a given criterion ore objective function is minimized.

As process model we consider the continuous linear system

$$\dot{x} = Ax + Bu + Cr, \tag{4.97}$$

$$y = Dx. \tag{4.98}$$

Note that we have included the term Cr in the state space model. Normally, we have $C = 0$ in connection with standard feedback systems. We will later in Section 4.8 show that it may be practical to use a model with $C \neq 0$ in case that we want integral action in the closed loop controlled system.

the reason for the term Cr is that a standard process model $\dot{x} = Ax + Bu$ and $y = Dx$ augmented with an integrator $\dot{z} = r - y$ for the controller results in a model of the type (4.97). We will discuss this later. However, note that the development will be more general if we work with the term Cr in the process model.

Let us define the deviation between the output y and the reference r by

$$e = r - y = r - Dx. \tag{4.99}$$

We are choosing an Linear Quadratic (LQ) criterion where the deviation defined by (4.99) is weighted, i.e.,

$$J = \frac{1}{2}e^T(t_1)Se(t_1) + \frac{1}{2}\int_{t_0}^{t_1}[e^TQe + u^TPu]dt. \tag{4.100}$$

Substituting for e gives

$$\begin{aligned} J = &\tfrac{1}{2}(r(t_1) - Dx(t_1))^TS(r(t_1) - Dx(t_1)) \\ &+\tfrac{1}{2}\int_{t_0}^{t_1}[(r - Dx)^TQ(r - Dx) + u^TPu]dt. \end{aligned} \tag{4.101}$$

where S, Q and $P > 0$ is symmetric weighting matrices.

We will now solve this problem of minimizing (4.101) subject to the process model (4.97) and (4.98) by using the maximum principle. We first form the Hamiltonian matrix,

$$H = \frac{1}{2}[(r - Dx)^TQ(r - Dx) + u^TPu] + p^T(Ax + Bu + Cr). \tag{4.102}$$

This can be expressed as follows

$$H = \frac{1}{2}(r^TQr - r^TQDx - x^TD^TQr + x^TD^TQDx + u^TPu) + p^T(Ax + Bu + Cr) \tag{4.103}$$

The Hamiltonian function is a scalar function. hence, we may write

$$H = \frac{1}{2}(r^TQr - 2r^TQDx + x^TD^TQDx + u^TPu) + p^T(Ax + Bu + Cr). \tag{4.104}$$

The optimal control is found by putting the gradient of H with respect to u equal to zero, i.e.,

$$\frac{\partial H}{\partial u} = Pu + B^T p = 0. \tag{4.105}$$

This gives

$$u = -P^{-1}B^T p. \tag{4.106}$$

In the same way as for optimal feed forward control from disturbances, v, we may show that the impulse vector, p, may be expressed as a linear function in the state vector, x, and of an at this stage unknown vector function h. The function h may be viewed as a feed forward function du to the external reference vector r. We have

$$p = Rx + h. \tag{4.107}$$

The optimal control is then given by

$$u = -P^{-1}B^T Rx - P^{-1}B^T h. \tag{4.108}$$

In order to use this solution we have to find expressions for R and h. By taking the time derivatives of Equation (4.107) we find

$$\dot{p} = \dot{R}x + R\dot{x} + \dot{h}. \tag{4.109}$$

Let us now obtain the equations for $\dot{p}$ and $\dot{x}$ and using those in (4.109). From the maximum principle we have that

$$\begin{aligned}\dot{p} &= -\tfrac{\partial H}{\partial x} = -D^T QDx - A^T p + D^T Qr \\ &= -D^T QDx - A^T Rx - A^T h + D^T Qr.\end{aligned} \tag{4.110}$$

Putting the optimal control into the state space model, Equation (4.97), we find

$$\dot{x} = Ax - BP^{-1}B^T Rx - BP^{-1}B^T h + Cr. \tag{4.111}$$

Putting (4.110) and (4.111) into (4.109) gives

$$\begin{aligned}&-D^T QDx - A^T Rx - A^T h + D^T Qr \\ &= \dot{R}x + R(Ax - BP^{-1}B^T Rx - BP^{-1}B^T h + Cr) + \dot{h}.\end{aligned} \tag{4.112}$$

This can be rearranged as follows

$$\begin{aligned}&(\dot{R} + A^T R + RA - RBP^{-1}B^T R + D^T QD)x \\ &+\dot{h} + (A - BP^{-1}B^T R)^T h - D^T Qr + RCr.\end{aligned} \tag{4.113}$$

This equation must hold for arbitrarily $x \neq 0$, and also recognize the Riccati equation. We may therefore write

$$-\dot{R} = A^T R + RA - RBP^{-1}B^T R + D^T QD, \tag{4.114}$$

$$-\dot{h} = (A - BP^{-1}B^T R)^T h - D^T Qr + RCr. \tag{4.115}$$

The final value boundary conditions for those differential equations are found as follows. From the maximum principle, Equation (4.7) we obtain

$$p(t_1) = \frac{\partial}{\partial x}\left[\tfrac{1}{2}(r(t_1) - Dx(t_1))^T S(r(t_1) - Dx(t_1))\right]_{t_1}. \tag{4.116}$$

This can be written as follows

$$p(t_1) = \frac{\partial}{\partial x}\left[\tfrac{1}{2}(r(t_1)^T Sr(t_1) - 2r(t_1)^T SD^T x(t_1) + x(t_1)^T D^T SDx(t_1))\right]_{t_1}. \tag{4.117}$$

Derivation of (4.117) of time gives

$$p(t_1) = D^T SDx(t_1) - D^T Sr(t_1). \tag{4.118}$$

Letting $t = t_1$ in (4.107) gives

$$p(t_1) = R(t_1)x(t_1) + h(t_1). \tag{4.119}$$

This must hold for arbitrarily final states $x(t_1)$. Hence, by comparing Equations (4.118) and (4.119) gives the boundary conditions

$$R(t_1) = D^T SD, \tag{4.120}$$

$$h(t_1) = -D^T Sr(t_1). \tag{4.121}$$

Note that we get the special case $R(t_1) = 0$ and $h(t_1) = 0$ if we do not weight the deviation e at the end time, ie set $S = 0$ in the optimal criterion. However, in this case there should be little reason to set $S = 0$. A better choice is to put $S = Q$. The reason for this is in order not to obtain bad tracking of the reference at $t = t_1$.

We see that equation (4.114) is the common Riccati equation. The only difference from before is that we now have a weighting matrix $D^T QD$ for the process state vector x. Both the Riccat equation and the equation (4.115) can be solved backwards in time from end-time t_1. Remember that we need the solutions at the present time t, ie. $R(t)$ and $h(t)$.

The optimal solution consists, as we have shown, of a feedback from the process state vector x as well as a feed-forward link from h. We see that the forward feedback does not affect the feedback. The vector h is determined by the reference vector r as well as by the feedback.

4.7.1 Conclusion

We summarize the results in the following theorem

Theorem 4.7.1 (Continuous time Linear Quadratic (LQ) optimal tracking)

Given a linear state space model $\dot{x} = Ax + Bu$ and $y = Dx$ as well a LQ optimal criterion as given in equation (4.101).

The optimal control that minimizes the optimal criterion is given by

$$u = G_1 x - P^{-1} B^T h, \tag{4.122}$$

where

$$G_1 = -P^{-1}B^T R(t), \tag{4.123}$$

and

$$\begin{aligned} -\dot{R} &= A^T R + RA - RBP^{-1}B^T R + D^T QD, \quad R(t_1) = D^T SD, & (4.124)\\ -\dot{h} &= (A + BG_1)^T h - D^T Qr + RCr, \quad h(t_1) = -D^T Sr(t_1). & (4.125)\end{aligned}$$

$\triangle$

Theorem 4.7.2 (Continuous time LQ optimal tracking: minimum of the objective)

Given the solution to the optimal tracking problem in theorem 4.7.1. The minimum value of the criterion over the time horizon $[t, t_1]$ is then given by

$$J(t) = \frac{1}{2}x^T Rx + x^T h + w, \tag{4.126}$$

where the time varying signal w is given by

$$-\dot{w} = \frac{1}{2}r^T Qr - \frac{1}{2}h^T BP^{-1}B^T h, \tag{4.127}$$

with boundary conditions for w at the final time t_1 given by

$$w(t_1) = \frac{1}{2}r^T(t_1)Sr(t_1). \tag{4.128}$$

$\triangle$

Let's study the steady state solution. This is what we get if the time horizon is infinite, i.e. $t_1 \to \infty$. If we put $\dot{h} = 0$ in equation (4.115) we find

$$h = (A - BP^{-1}B^T R)^{-T}(D^T Q - RC)r. \tag{4.129}$$

The optimal control is thus given by

$$u = G_1 x + G_2 r, \tag{4.130}$$

where

$$\begin{aligned} G_1 &= -P^{-1}B^T R, & (4.131)\\ G_2 &= -P^{-1}B^T(A - BP^{-1}B^T R)^{-T}(D^T Q - RC), & (4.132)\end{aligned}$$

where R is the solution of the algebraic Riccati equation. The optimal control in this case is given by a constant feedback from x and a constant feed-forward control from the reference r.

Example 4.1 (Optimalt følgesystem)
Given a process described with a SISO model with one state, with model

$$\begin{aligned} \dot{x} &= -0.5x + u, & (4.133)\\ y &= x, & (4.134)\end{aligned}$$

with initial condition $x(t_0) = 0$.

the output y *should follow a given reference signal* $r(t)$. *We chose the following optimal objective*

$$J = \frac{1}{2}s(r(t_1) - y(t_1)^2 + \frac{1}{2}\int_{t_0}^{t_1}(q(r-y)^2 + pu^2)dt, \tag{4.135}$$

where s, q *and* p *are scalar weighting parameters.*

The optimal control u *which minimizes the objective* J *is given by*

$$u = g_1 x + g_2 h, \tag{4.136}$$

where

$$g_1 = -\frac{R}{p}, \quad g_2 = -\frac{h}{p}. \tag{4.137}$$

R *is the solution to the Riccati equation and the feed-forward control* h *is given by*

$$-\dot{R} = -R - \frac{R^2}{p} + q, \tag{4.138}$$

$$-\dot{h} = -(\frac{1}{2} + \frac{R}{p})h - qr. \tag{4.139}$$

the boundary conditions are given by Eqs. (4.120) and (4.121).

$$R(t_1) = s, \tag{4.140}$$
$$h(t_1) = -sr(t_1). \tag{4.141}$$

The steady state solution of the Riccati equation as well as the stationary feedback are given by

$$R = p\frac{\sqrt{1+4\frac{q}{p}} - 1}{2}, \tag{4.142}$$

$$g_1 = \frac{\sqrt{1+4\frac{q}{p}} - 1}{2}. \tag{4.143}$$

Note that if we set the boundary condition $R(t_1)$ *for the Riccati equation (4.138) similar to the stationary solution, i.e.* $R(t_1) = R$ *which means that* $s = R$, *the solution of the dynamic Riccati equation will be constant for all time* t_0 *leqt leq*t_1. *In this case, this means that the deviation* $r-y$ *at the end time is weighted by* $s = R$. *This solution to the problem is especially simple because the optimum on the draw consists of a constant feedback and a dynamic feed-forward. See Figures 4.1 and 4.2 for simulations.*

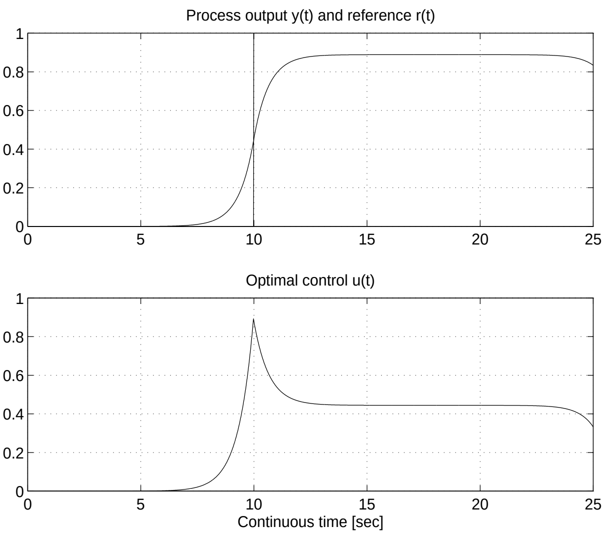

Figure 4.1: The figure shows simulations of y and u for example 4.1. We have used the weights $s = 2$, $q = 2$ and $p = 1$. We see that the output y reacts before the jump in the reference at $t = 10$. This is typical of optimal tracking systems (and for predictive control) because one in advance, know the future change in the reference.

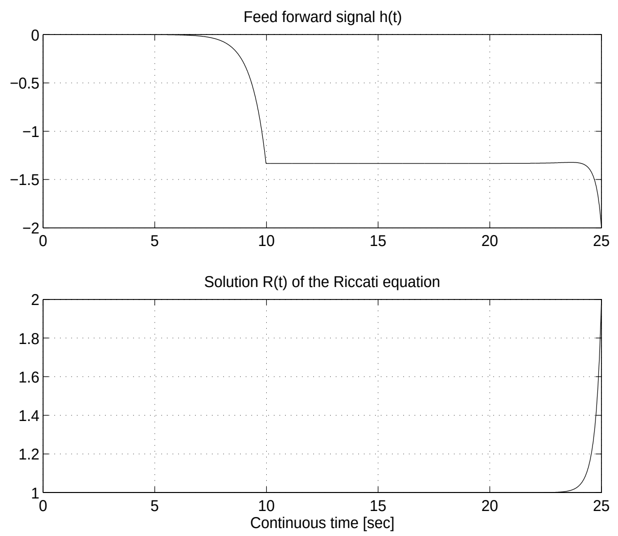

Figure 4.2: The figure shows simulations of R and h for example 4.1. R is the solution of the dynamic Riccati equation. h is the optimal feed-forward signal. We have used the weights $s = 2$, $q = 2$ and $p = 1$.

4.8 LQ optimal tracking system with prediction and integral action

A standard Linear quadratic (LQ) optimal tracking system will generally have a stationary deviation between the reference and the output vector (which must follow the reference). In this section we will study a method to eliminate the stationary deviated (including integral action).

In this section, we will show how we can expand the results which was derived in section 4.7 so that the LQ optimal tracking system has integral action.

We can call the results in section 4.7 a standard wording and solving the follow-up problem. The technique we are going to use here is to include a model of the integral effect in the process model and the criterion. We can then set this to standard form. The solution is further given as in section 4.7.

4.8.1 Augmented process and controller model

Assume given a linear state space model for the system

$$\dot{x} = Ax + Bu, \tag{4.144}$$

$$y = Dx. \tag{4.145}$$

A method used to achieve integral action in optimal systems is to include the derivative of the controller error $r - y$ in the model. we define

$$\dot{z} = r - y = r - Dx, \tag{4.146}$$

where we have used Eq. (4.145). We have here introduced a state z which is given by integration of (4.146). We combine (4.146) with the process model (4.144) and (4.145) and obtain

$$\begin{bmatrix} \dot{x} \\ \dot{z} \end{bmatrix} = \overbrace{\begin{bmatrix} A & 0 \\ -D & 0 \end{bmatrix}}^{\tilde{A}} \overbrace{\begin{bmatrix} x \\ z \end{bmatrix}}^{\tilde{x}} + \overbrace{\begin{bmatrix} B \\ 0 \end{bmatrix}}^{\tilde{B}} u + \overbrace{\begin{bmatrix} 0 \\ I \end{bmatrix}}^{C_1} r. \tag{4.147}$$

In the same way, the output vector (4.145) can be written as

$$y = \overbrace{\begin{bmatrix} D & 0 \end{bmatrix}}^{D_1} \overbrace{\begin{bmatrix} x \\ z \end{bmatrix}}^{\tilde{x}}. \tag{4.148}$$

This gives the augmented (combined) state space model

$$\dot{\tilde{x}} = \tilde{A}\tilde{x} + \tilde{B}u + C_1 r, \tag{4.149}$$

$$y = D_1 \tilde{x}. \tag{4.150}$$

where the model matrices and vectors are defined as in (4.147) and (4.148).

For later use, we define a slightly different formulation in the same way of the extended state space model (4.148) and (4.150).

$$\dot{\tilde{x}} = \tilde{A}\tilde{x} + \tilde{B}u + \tilde{C}\tilde{r}, \tag{4.151}$$

$$\tilde{y} = \tilde{D}\tilde{x}, \tag{4.152}$$

where

$$\tilde{y} = \begin{bmatrix} y \\ z \end{bmatrix}, \quad \tilde{D} = \begin{bmatrix} D & 0 \\ 0 & I \end{bmatrix}, \quad \tilde{C} = \begin{bmatrix} 0 & 0 \\ I & 0 \end{bmatrix}, \quad \tilde{r} = \begin{bmatrix} r \\ 0 \end{bmatrix}. \tag{4.153}$$

4.8.2 Formulating the objective

From the theory of LQ optimal systems (section 4.2), 4.7)) we know that the optimal on the move among other things, consists of a feedback from the whole the state vector of the process. The reason for this is that the conditions must be weighted in the criterion such that all conditions are observable (possibly detectable) seen from the criterion. To ensure that all conditions are observable from the criterion it is natural to emphasize the regulator state vector z in addition to emphasizing the deviation $r - y$. It would therefore be natural to choose a criterion of the form

$$J = \frac{1}{2}[(r-y)^T S(r-y) + z^T S_z z]_{t_1} + \frac{1}{2}\int_{t_0}^{t_1} [(r-y)^T Q(r-y) + z^T Q_z z + u^T P u] dt. \tag{4.154}$$

We will now show that this criterion can be set to standard form, i.e. in the same form as the criterion used in connection with optimal tracking. See section 4.7.

Let's start with based on the definitions given in (4.153) and looks at the discrepancy between $\tilde{r}$ and $\tilde{y}$. We have

$$\tilde{r} - \tilde{y} = \tilde{r} - \tilde{D}\tilde{x} = \overbrace{\begin{bmatrix} r \\ 0 \end{bmatrix}}^{\tilde{r}} - \overbrace{\begin{bmatrix} D & 0 \\ 0 & I \end{bmatrix}}^{\tilde{D}} \overbrace{\begin{bmatrix} x \\ z \end{bmatrix}}^{\tilde{x}} = \begin{bmatrix} r - Dx \\ -z \end{bmatrix}. \tag{4.155}$$

We define the augmented weighting matrix

$$\tilde{Q} = \begin{bmatrix} Q & 0 \\ 0 & Q_z \end{bmatrix}. \tag{4.156}$$

We then have that

$$\begin{aligned} &(\tilde{r} - \tilde{D}\tilde{x})^T \tilde{Q} (\tilde{r} - \tilde{D}\tilde{x}) \\ &= \begin{bmatrix} (r - Dx)^T & z^T \end{bmatrix} \begin{bmatrix} Q & 0 \\ 0 & Q_z \end{bmatrix} \begin{bmatrix} r - Dx \\ z \end{bmatrix} \\ &= (r - Dx)^T Q (r - Dx) + z^T Q_z z. \end{aligned} \tag{4.157}$$

Let us further define the following weighting matrix for the error $\tilde{r} - \tilde{D}\tilde{x}$ at the final time instant t_1

$$\tilde{S} = \begin{bmatrix} S & 0 \\ 0 & S_z \end{bmatrix}. \tag{4.158}$$

This means that the criterion (4.154) may be written equivalently as

$$J = \frac{1}{2}[(\tilde{r} - \tilde{D}\tilde{x})^T \tilde{S} (\tilde{r} - \tilde{D}\tilde{x})]_{t_1} + \frac{1}{2}\int_{t_0}^{t_1} [(\tilde{r} - \tilde{D}\tilde{x})^T \tilde{Q} (\tilde{r} - \tilde{D}\tilde{x}) + u^T P u] dt \tag{4.159}$$

We notice for lather use that this objective is of the same form as the objective (4.100).

4.8.3 Solution to the optimal tracking problem with integral action

We see that the criterion given by (4.159) is of the same form as the criterion (4.100). Furthermore, the state space model given by (4.152) and (4.153) is of the same form as the state space model (4.97) and (4.98). This means that we can use the same solution as derived in section 4.8 and as presented in theorem 4.7.1.

The only practical difference in the solution in section 4.8 and theorem 4.7.1 is that we have got two new folding matrices Q_z and S_z that emphasize the integrator state vector z. In addition, the dimension of the Riccati equation has increased and thus the complexity of the problem. On the other hand, there are many zeros in the expanded matrices we has defined. It may therefore be useful to rewrite the solution.

We end the discussion by concluding that the general solution The problem with integral effect is quite equivalent to the solution presented in theorem 4.7.1, but replaced with extended model matrices and vectors as presented above. We therefore do not repeat the solution. In the next section, however, we will study a suboptimal solution.

4.8.4 Suboptimal solution

In this section, we are by problem definition interested in zero stationary deviation between the reference r and the output vector y. In order to be able to analyze the stationary deviation we mathematically let $t \to \infty$. This means that it is reasonable that we only study optimal tracking systems with integral effect for time horizons of a certain size. If the time horizon is chosen small, it can in many cases be impossible to achieve zero steady state error. You can also have problems with stability. We disregard the case of constantly sliding horizons, i.e. *receding horizon* problems.

It is therefore reasonable to assume a large time horizon. By large we mean here that the time horizon is greater than the dominant one the time constant of the feedback system. We can therefore study the stationary Riccati equation. This means that we also do not need the boundary conditions for the dynamic matrix Riccati equation. We also have a great advantage because we are guaranteed that the closed system is stable. This is of great practical interest.

From theorem 4.7.1 we have that the optimum p aa drag is given by

$$u = G_1\tilde{x} - P^{-1}\tilde{B}^T h. \tag{4.160}$$

This means that the pull is given by a feedback from the process state vector x and a feedback from the integrator state z. We see this by spliting up (4.160. We have

$$u = G_x x + G_z z - P^{-1}\tilde{B}^T h. \tag{4.161}$$

where G_x and G_z are submatrices from G_1. We assume a large time horizon. The feedback gain matrix G_1 is therefore given by

$$G_1 = -P^{-1}\tilde{B}^T R, \tag{4.162}$$

where R is given by steady state solution to the ARE (the Algebraic Riccati Equation) in Theorem 4.7.1. We have

$$\tilde{A}^T R + R\tilde{A} - R\tilde{B}P^{-1}\tilde{B}^T R + \tilde{D}^T\tilde{Q}\tilde{D} = 0. \tag{4.163}$$

We notice that

$$\tilde{D}^T\tilde{Q}\tilde{D} = \begin{bmatrix} D^T QD & 0 \\ 0 & Q_z \end{bmatrix}. \tag{4.164}$$

G_1 og R beregnes enkelt for eksempel med MATLAB funksjonene **lqr** eller **lqr2**.

Let's study the boundary conditions for the dynamic equation for calculating the forward switching signal $h(t)$. We have assumed a large time horizon so that we can use the stationary solution to the Riccati equation to determine a constant feedback matrix G_1. In section 4.4 we gave a justification that it has no meaning to weight the end state if the horizon is infinite and that we thereby can set $S = 0$. Actually, S is arbitrary in this case because S is not entering in the stationary Riccati equation.

Against this background, it will in our case be tempting to use $\tilde{S} = 0$ so that the boundary condition becomes $h(t_1) = 0$. On the other hand, we want zero stationary offsets. A boundary condition $h(t_1) = 0$ will generally make it impossible to meet the requirement for zero stationary deviation at the end time. The reason for this is of course that the forwarding signal h go to zero when we reach the end time. This means that you do not have stationary conditions near the end time.

Let's look at the case that $\tilde{S} \neq 0$. From theorem 4.7.1 we have that

$$h(t_1) = -\tilde{D}\tilde{S}\tilde{r}(t_1) = -D_1^T Sr(t_1) = \begin{bmatrix} -D^T Sr(t_1) \\ 0 \end{bmatrix}, \tag{4.165}$$

where we have used the definitions for $\tilde{S}$ as defined in (4.158), $\tilde{D}$ and $\tilde{r}$ as defined in (4.153) and D_1 as defined in (4.148).

We see from this that if $S \neq 0$ the forward switch will be actived also at the end time. It can be shown that this does not necessarily give zero stationary deviation at end-time.

To achieve zero stationary deviation also at the end time we can use the stationary solution of the dynamic equation (4.125) as boundary condition. We then have the following limit condition for h.

$$h(t_1) = A_{cl}^{-T}(\tilde{D}^T\tilde{Q} - R\tilde{C})\tilde{r}(t_1), \tag{4.166}$$

$$A_{cl} = \tilde{A} + \tilde{B}R, \tag{4.167}$$

where R is the solution to the steady state Riccati equation (4.149). The boundary conditions may be written more simplified as follows

$$h(t_1) = A_{cl}^{-T}(D_1^T Q - RC_1)r(t_1). \tag{4.168}$$

We notice that the fold matrix Q_z which limits (emphasizes) z is not included in the calculation of h.

Such a suboptimal solution as discussed above will of course generally give one higher value on the criterion J compared to the optimal solution. This is the price you have to pay for using a constant feedback matrix G_1 and in addition have zero steady state deviation also at the end-time.

Example 4.2 (Optimal tracking system with integral action)
In example 4.1 we got stationary deviation between the reference r and the output y. In this example we will use the theory described in this section and the same process as in example 4.1 and show that the stationary deviation becomes zero. Given a process as described in example 4.1. The controller's integrator state z is defined by

$$\dot{z} = r - x, \tag{4.169}$$

Combining this with the model described in example 4.1 we obtain

$$\begin{bmatrix} \dot{x} \\ \dot{z} \end{bmatrix} = \overbrace{\begin{bmatrix} -0.5 & 0 \\ -1 & 0 \end{bmatrix}}^{\tilde{A}} \overbrace{\begin{bmatrix} x \\ z \end{bmatrix}}^{\tilde{x}} + \overbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}^{\tilde{B}} u + \overbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}^{C_1} r, \tag{4.170}$$

$$y = \overbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}^{\tilde{D}} \begin{bmatrix} x \\ z \end{bmatrix}. \tag{4.171}$$

The output y should follow a given reference $r(t)$. We therefore choose the following optimal criterion

$$J = \frac{1}{2}[S(r(t_1) - y(t_1))^2 + S_z z^2]_{t_1} + \frac{1}{2}\int_{t_0}^{t_1} [Q(r-y)^2 + Q_z z^2 + Pu^2]dt, \tag{4.172}$$

where we have selected $S = 2$, $S_z = 1$, $Q = 2$, $Q_z = 1$ and $P = 1$ are scalar weights. The station solution can be found, for example, by using the "MATLAB Control System Toolbox" function, $[-G_1, R] = lqr2(\tilde{A}, \tilde{B}, \tilde{Q}, P)$. where

$$\tilde{Q} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}. \tag{4.173}$$

This gives

$$R = \begin{bmatrix} 1.562 & -1.000 \\ -1.000 & 2.062 \end{bmatrix}, \quad G_1 = \begin{bmatrix} -1.562 & 1.000 \end{bmatrix} \tag{4.174}$$

This gives the control

$$u = -1.5616x + z - \overbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}^{-P^{-1}\tilde{B}^T} \overbrace{\begin{bmatrix} h_1 \\ h_2 \end{bmatrix}}^{h} = -1.5616x + z - h_1(t), \tag{4.175}$$

where h is the solution of the dynamic equation (4.125) and z is given by (4.169). We have used (4.168) as a boundary condition for equation (4.125). The simulation results shown in figure 4.3 show that we have zero stationary offset/deviation. This was not the case in example 4.1.

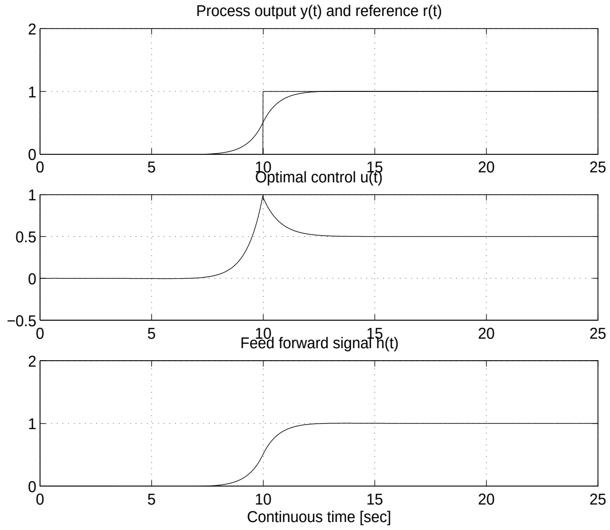

Figure 4.3: The figure shows simulations of y, u and h for example 4.2. We have used the weights $S = 2$, $Q = 2$, $Q_z = 1$ and $P = 1$. The start and end times are $t_0 = 0$ and $t_1 = 25$, respectively. We have used (4.166) as the boundary condition for $h(t_1)$. We see that the output y reacts before the jump in the reference at $t = 10$. This is typical of optimal tracking systems (and for predictive control) because one know the future change in the reference. We see that we have zero stationary offset/deviation between r and y. This was not the case in example 4.1 and the figures 4.1 and 4.2.

4.9 Weighting control rate of change $\dot{u}$

4.9.1 Standard LQ optimal control with weighting the control rate of change $\dot{u}$

Consider a LQ objective of the form

$$J = \frac{1}{2}x(t_1)^T Sx(t_1) + \frac{1}{2}\int_{t_0}^{t_1}(x^T Qx + u^T Pu + \dot{u}^T \mathcal{R}\dot{u})dt, \tag{4.176}$$

where we in addition to emphasizing the state vector, x, and on the control vector, u, emphasizes the derivative of the control i.e., $\dot{u}$. The advantage of this is that we can add weight on the speed of the control action via the weight matrix $\mathcal{R}$. We can thus by increasing $\mathcal{R}$ achieve a smoother/softer and calmer control action u. This is appropriate in systems where we do not want rapid changes in the control action.

This problem can be solved by transforming the problem into a standard LQ problem. Let's introduce a new control $\tilde{u}$ such that

$$\dot{u} = \tilde{u}. \tag{4.177}$$

We consider this as a state equation with u as the state. We can then set up an extended state space model

$$\begin{bmatrix} \dot{x} \\ \dot{u} \end{bmatrix} = \begin{bmatrix} A & B \\ 0_{r\times n} & 0_{r\times r} \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} + \begin{bmatrix} 0_{n\times r} \\ I_{r\times r} \end{bmatrix} \tilde{u}. \tag{4.178}$$

ie. such that we have an extended state space model of the form

$$\dot{\tilde{x}} = \tilde{A}\tilde{x} + \tilde{B}\tilde{u}, \tag{4.179}$$

where

$$\tilde{x} = \begin{bmatrix} x \\ u \end{bmatrix}, \quad \tilde{A} = \begin{bmatrix} A & B \\ 0_{r\times n} & 0_{r\times r} \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} 0_{n\times r} \\ I_{r\times r} \end{bmatrix}. \tag{4.180}$$

The LQ objective may then be written on standard form as

$$J = \frac{1}{2}\tilde{x}(t_1)^T \tilde{S}\tilde{x}(t_1) + \frac{1}{2}\int_{t_0}^{t_1} (\tilde{x}^T \tilde{Q}\tilde{x} + \tilde{u}^T \mathcal{R}\tilde{u})dt, \tag{4.181}$$

where the weighting matrices are given by

$$\tilde{Q} = \begin{bmatrix} Q & 0_{n\times r} \\ 0_{r\times n} & P \end{bmatrix}, \quad \tilde{S} = \begin{bmatrix} S & 0_{n\times r} \\ 0_{r\times n} & 0_{r\times r} \end{bmatrix}. \tag{4.182}$$

The optimal control $\tilde{u}$ is then given by

$$\tilde{u} = G\tilde{x} \tag{4.183}$$

$$G = -\mathcal{R}^{-1}\tilde{B}^T R, \tag{4.184}$$

where R is the positive definite solution ti the Riccati equation

$$-\dot{R} = \tilde{A}^T R + \tilde{A}R - R\tilde{B}\mathcal{R}\tilde{B}^T R + \tilde{Q}, \tag{4.185}$$

with boundary conditions at the final time instant t_1, i.e.

$$R(t_1) = \tilde{S}. \tag{4.186}$$

Remark that we know have obtained an equation

$$\dot{u} = G_1 x + G_2 u \tag{4.187}$$

which has to be solved with respect to the control action u so we can put this control to the process. This can usually be easiest done by discretization. We should note that there is one discrete variant of this problem that we will discuss in the section on optimal regulation of discrete time systems.

4.10 Specified final state and open loop control

The control objective to be studied in this section is to drive the state $x(t)$ in a linear system $\dot{x} = Ax + Bu$ from an initial state $x(t_0)$ to a final state $x(t_1)$ using minimum control energy. The initial state $x(t_0)$ is known and the final state $x(t_1)$ is specified.

This optimal control problem can be solved by minimizing a quadratic performance index. Since $x(t_1)$ is specified it is redundant to include a final state weighting in the cost index (performance index). Hence, it make sense to let the final state weighting matrix $S = 0$. In order to simplify the solution, let $Q = 0$ also.[1] The resulting quadratic performance index is given by

$$J = \frac{1}{2}\int_{t_0}^{t_1} u^T P u dt. \tag{4.188}$$

Note that $u = 0 \; \forall \; t \in [t_0, t_1 >$ gives a minimum $J = 0$ when $P > 0$. However, this control does in general not drive the state to the specified final state $x(t_1)$. Hence, $u = 0$ is not a solution to our problem.

We will solve this optimal control problem by using the maximum principle. The Hamilton function is given by

$$H = \frac{1}{2}u^T P u + p^T(Ax + Bu). \tag{4.189}$$

The optimal control is determined from the condition $\frac{\partial H}{\partial u} = 0$, i.e.,

$$u = -P^{-1}B^T p, \tag{4.190}$$

where we have assumed that $P > 0$. Substituting the optimal control into the state and costate equations gives

$$\dot{x} = Ax - BP^{-1}B^T p, \tag{4.191}$$

$$\dot{p} = -A^T p. \tag{4.192}$$

As we can see, the choice $Q = 0$ has decoupled the costate equation from the state equation. Hence, the solution of the costate equation is simply

$$p(t) = e^{-A^T(t-t_1)}p(t_1) = e^{A^T(t_1-t)}p(t_1), \tag{4.193}$$

where, at this stage, $p(t_1)$ is unknown. Substituting this into the state Equation (4.191) gives

$$\dot{x} = Ax - BP^{-1}B^T e^{A^T(t_1-t)}p(t_1). \tag{4.194}$$

The solution of the state equation with the optimal control is given by

$$x(t) = e^{A(t-t_0)}x(t_0) - (\int_{t_0}^{t} e^{A(t-\tau)}BP^{-1}B^T e^{A^T(t_1-\tau)}d\tau)p(t_1), \tag{4.195}$$

[1] In fact, as we will show, this problem has an analytical solution.

We can now find $p(t_1)$ from the equation obtained by evaluating (4.195) for $t = t_1$. Putting $t = t_1$ in (4.195) gives

$$x(t_1) = e^{A(t_1-t_0)}x(t_0) - W_c(t_0, t_1)p(t_1), \tag{4.196}$$

where

$$W_c(t_0, t_1) = \int_{t_0}^{t_1} e^{A(t_1-\tau)}BP^{-1}B^T e^{A^T(t_1-\tau)}d\tau, \tag{4.197}$$

is defined as the *weighted controllability gramian*. The gramian is weighted because it depends upon the control weighting matrix P. Note that the *weighted controllability gramian* reduces to the standard *controllability gramian* when $P = I$ and $t_0 = 0$.

We have from (4.196) that the final costate is given by

$$p(t_1) = -W_c(t_0, t_1)^{-1}(x(t_1) - e^{A(t_1-t_0)}x(t_0)), \tag{4.198}$$

provided $W_c(t_0, t_1)$ is non-singular. The costate $p(t)$ is then given by (putting (4.198) into (4.193) gives)

$$p(t) = -e^{A^T(t_1-t)}W_c(t_0, t_1)^{-1}(x(t_1) - e^{A(t_1-t_0)}x(t_0)). \tag{4.199}$$

Substituting this into the expression for the optimal control, i.e. $u = -P^{-1}B^T p$, gives the optimal control

$$u(t) = P^{-1}B^T e^{A^T(t_1-t)}W_c(t_0, t_1)^{-1}(x(t_1) - e^{A(t_1-t_0)}x(t_0)). \tag{4.200}$$

if $W_c(t_0, t_1)$ is non-singular. Note that the optimal control (4.200) for single input systems is independent of the control weighting p. Since $u(t)$ is defined in terms of the inverse of the gramian $W_c(t_0, t_1)$ the optimal control exists for arbitrary $x(t_0)$ and $x(t_1)$ if and only if $\det(W_c(t_0, t_1)) \neq 0$. This corresponds to controllability of the plant. This means that if the system (A, B) is controllable then there exists a minimum-energy control to drive any $x(t_0)$ to any desired $x(t_1)$.

The control (4.200) is an open-loop control since $u(t)$ does not depend on the current state $x(t)$. It depends only on the initial and the final states (and time), and it can be precomputed and then applied for all t in $[t_0, t_1]$.

4.10.1 On the controllability gramian

Definition 4.1 (Weighted controllability gramian) *The weighted controllability gramian for the system (A, B) is defined as*

$$W_c(t_0, t) = \int_{t_0}^{t} e^{A(t-\tau)}BP^{-1}B^T e^{A^T(t-\tau)}d\tau \tag{4.201}$$

$$= \int_{0}^{t-t_0} e^{A\tau}BP^{-1}B^T e^{A^T\tau}d\tau, \tag{4.202}$$

where P is a non-singular weighting matrix.

Note that the gramian $W_c(t_0, t)$ only is dependent on the difference $t - t_0$. This means that $W_c(0, t - t_0) = W_c(t_0, t)$. This is the reason for the short-hand notation $W_c(t_0, t) = W_c(t - t_0)$ which sometimes is used.

It is useful to recognize the relationship between the gramian $W_c(t_0, t)$ and the solution of a matrix Lyapunov equation. We have the following proposition.

Proposition 4.1 *The weighted controllability gramian $W_c(t_0, t)$ can be computed from the solution of the Lyapunov matrix differential equation*

$$\dot{W} = AW + WA^T + BP^{-1}B^T \tag{4.203}$$

which has the solution

$$\begin{aligned} W(t) &= e^{A(t-t_0)}W(t_0)e^{A^T(t-t_0)} + \int_{t_0}^{t} e^{A(t-\tau)}BP^{-1}B^Te^{A^T(t-\tau)}d\tau \\ &= e^{A(t-t_0)}W(t_0)e^{A^T(t-t_0)} + \int_{0}^{t-t_0} e^{A\tau}BP^{-1}B^Te^{A^T\tau}d\tau. \end{aligned} \tag{4.204}$$

If the initial condition is zero, i.e., $W(t_0) = 0$, then $W_c(t_0, t) = W(t)$.

Proof

The time derivative of (4.204) is

$$\begin{aligned} \dot{W}(t) &= Ae^{A(t-t_0)}W(t_0)e^{A^T(t-t_0)} + e^{A(t-t_0)}W(t_0)e^{A^T(t-t_0)}A^T \\ &+ e^{A(t-t_0)}BP^{-1}B^Te^{A^T(t-t_0)}. \end{aligned} \tag{4.205}$$

Substituting (4.205) and (4.204) into (4.203) gives

$$e^{A(t-t_0)}BP^{-1}B^Te^{A^T(t-t_0)} =$$
$$\int_{t_0}^t Ae^{A(t-\tau)}BP^{-1}B^Te^{A^T(t-\tau)}d\tau + \int_{t_0}^t e^{A(t-\tau)}BP^{-1}B^Te^{A^T(t-\tau)}A^Td\tau + BP^{-1}B^T$$

and

$$e^{A(t-t_0)}BP^{-1}B^Te^{A^T(t-t_0)} = -\int_{t_0}^{t} \frac{d}{d\tau}\left[e^{A(t-\tau)}BP^{-1}B^Te^{A^T(t-\tau)}\right]d\tau + BP^{-1}B^T$$

and

$$e^{A(t-t_0)}BP^{-1}B^Te^{A^T(t-t_0)} = -\left[e^{A(t-\tau)}BP^{-1}B^Te^{A^T(t-\tau)}\right]_{t_0}^{t} + BP^{-1}B^T$$

which is true. Hence, (4.204) is a solution of (4.188). **QED**.

Remark 4.1 *Consider the solution (4.204) and (4.203) with initial condition $W(t_0) = 0$. Substituting (4.205) into (4.203) gives the matrix Lyapunov equation*

$$AW_c(t_0, t) + W_c(t_0, t)A^T = e^{A(t-t_0)}BP^{-1}B^Te^{A^T(t-t_0)} - BP^{-1}B^T \tag{4.206}$$

This is a linear equation which can be used to compute the gramian $W_c(t_0, t)$. This equation is frequently used when A is stable and $t \to \infty$. See e.g. the Control System Toolbox for MATLAB function $W_c = \mathrm{gram}(A, B)$. Equation (4.206) can be used for unstable A and finite t.

Remark 4.2 *It is important to note that (4.206) only can be used when the solution is unique or when (4.206) has a solution. Equation (4.206) can not be used on systems with* $A = 0$, $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ *and* $A = \begin{bmatrix} \lambda & 1 \\ 0 & -\lambda \end{bmatrix}$ *The reason for this is that the Lyapunov equation does not have a unique solution in these cases.*

Remark 4.3 *Note that the controllability gramian* W_c *((4.202) with* $P = I$*) is related to the controllability matrix* C_n *for the pair* (A, B) *as*

$$W_c(t_0, t) = C_n F(t) C_n^T \tag{4.207}$$

where $F(t)$ *is a matrix. The matrix* $F(t)$ *can be deduced by using the series equivalent to* $e^{A\tau}$ *and the Cayley-Hamilton theorem. Se example 4.7 for an illustration.*

4.10.2 Illustrating examples

Example 4.3 *Consider the system matrix*

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}. \tag{4.208}$$

Problem *Show that the transition matrix is given by*

$$e^{At} = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}. \tag{4.209}$$

Solution *The system matrix* A *is* nilpotent[2] *because* $A^2 = 0$*. This implies that the series expansion for* e^{At} *is finite, i.e.*

$$e^{At} = I + At. \tag{4.210}$$

Example 4.4 *Consider the system* $\dot{x} = Ax + Bu$ *with system matrices*

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{4.211}$$

Problem *Show that the weighted controllability gramian is given by*

$$W_c(t_0, t) = \frac{1}{p} \begin{bmatrix} \frac{(t-t_0)^3}{3} & \frac{(t-t_0)^2}{2} \\ \frac{(t-t_0)^2}{2} & t - t_0 \end{bmatrix}. \tag{4.212}$$

Solution *Integrating*

$$W_c(t_0, t) = \int_{t_0}^{t} e^{A(t-\tau)} B P^{-1} B^T e^{A^T(t-\tau)} d\tau. \tag{4.213}$$

with $P = p$ *as a scalar weight gives*

$$\begin{aligned} W_c(t_0, t) &= \frac{1}{p} \int_{t_0}^{t} \begin{bmatrix} 1 & t-\tau \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ t-\tau & 1 \end{bmatrix} d\tau \\ &= \frac{1}{p} \int_{t_0}^{t} \begin{bmatrix} (t-\tau)^2 & t-\tau \\ t-\tau & 1 \end{bmatrix} d\tau. \end{aligned} \tag{4.214}$$

[2] A nilpotent matrix is a matrix A where $A^k = 0$ for some k.

This gives

$$
\begin{aligned}
W_c(t_0,t) &= \frac{1}{p}\left|\begin{matrix} -\frac{1}{3}(t-\tau)^3 & -\frac{1}{2}(t-\tau)^2 \\ -\frac{1}{2}(t-\tau)^2 & \tau \end{matrix}\right|_{t_0}^{t} \\
&= \frac{1}{p}\begin{bmatrix} 0 & 0 \\ 0 & t \end{bmatrix} - \frac{1}{p}\begin{bmatrix} -\frac{1}{3}(t-t_0)^3 & -\frac{1}{2}(t-t_0)^2 \\ -\frac{1}{2}(t-t_0)^2 & t_0 \end{bmatrix} \\
&= \frac{1}{p}\begin{bmatrix} \frac{1}{3}(t-t_0)^3 & \frac{1}{2}(t-t_0)^2 \\ \frac{1}{2}(t-t_0)^2 & t-t_0 \end{bmatrix}. \qquad (4.215)
\end{aligned}
$$

As we see, the gramian $W_c(t_0,t)$ is only dependent on the difference $t-t_0$. This means that $W_c(0,t-t_0)=W_c(t_0,t)$.

Example 4.5 *The objective in this example is to compute the weighted controllability gramian $W_c(t_0,t)$ as in Example 4.4 but now by using the differential matrix Lyapunov equation approach as illustrated in (4.203) and (4.204). Let*

$$
W_c(t_0,t) = \begin{bmatrix} w_{11} & w_{21} \\ w_{21} & w_{22} \end{bmatrix}, \qquad (4.216)
$$

then (4.188) gives

$$
\dot{W} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} W + W \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + \overbrace{\begin{bmatrix} 0 & 0 \\ 0 & \frac{1}{p} \end{bmatrix}}^{BP^{-1}B^T}, \qquad (4.217)
$$

which gives the scalar differential equations

$$
\begin{aligned}
\dot{w}_{11} &= 2w_{21}, && (4.218)\\
\dot{w}_{21} &= w_{22}, && (4.219)\\
\dot{w}_{22} &= \frac{1}{p}. && (4.220)
\end{aligned}
$$

We can now integrating these equations by using zero initial conditions, i.e. $W(t=0)=0$, and from time t_0 to t. We have

$$
w_{22} = \frac{1}{p}\int_{t_0}^{t} d\tau = \frac{t-t_0}{p}. \qquad (4.221)
$$

Putting (4.221) into (4.219) gives

$$
w_{21} = \frac{1}{p}\int_{t_0}^{t} (\tau - t_0) d\tau = \frac{1}{2p}\left[\,(\tau-t_0)^2\,\right]_{t_0}^{t} = \frac{(t-t_0)^2}{2p} \qquad (4.222)
$$

and so on. Hence

$$
W_c(t,t_0) = \begin{bmatrix} w_{11} & w_{21} \\ w_{21} & w_{22} \end{bmatrix} = \begin{bmatrix} \frac{(t-t_0)^3}{3p} & \frac{(t-t_0)^2}{2p} \\ \frac{(t-t_0)^2}{2p} & \frac{t-t_0}{p} \end{bmatrix}. \qquad (4.223)
$$

Example 4.6 *An object obeying Newton's law satisfies*

$$\dot{x} = \overbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}^{A} x + \overbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}^{B} u \tag{4.224}$$

where $x = \begin{bmatrix} x_1 & x_2 \end{bmatrix}^T$ *with* x_1 *the position,* x_2 *the velocity and* u *an acceleration input.*

The control objective is to drive the state from an initial state $x(t_0)$ *to any final state* $x(t_1)$*, while minimizing the performance index*

$$J = \frac{1}{2} \int_{t_0}^{t_1} p u^2 dt. \tag{4.225}$$

The controllability gramian can be solved by using the definition or from (4.203) and (4.204) with zero initial conditions $W(t_0) = 0$*. Se Examples 4.4 and 4.5. From this we have*

$$W_c(t, t_0) = \begin{bmatrix} w_{11} & w_{21} \\ w_{21} & w_{22} \end{bmatrix} = \begin{bmatrix} \frac{(t-t_0)^3}{3p} & \frac{(t-t_0)^2}{2p} \\ \frac{(t-t_0)^2}{2p} & \frac{t-t_0}{p} \end{bmatrix}. \tag{4.226}$$

In order to compute the optimal control we need the inverse of $W_c(t_0, t_1)$*. We have*

$$W_c^{-1}(t_0, t_1) = \frac{12p}{(t_1 - t_0)^3} \begin{bmatrix} 1 & -\frac{t_1-t_0}{2} \\ -\frac{t_1-t_0}{2} & \frac{(t_1-t_0)^2}{3} \end{bmatrix}. \tag{4.227}$$

The optimal control is found by using (4.200). First, compute

$$\begin{aligned} B^T e^{A^T(t_1-t)} W_c^{-1}(t_0, t_1) &= \tfrac{12p}{(t_1-t_0)^3} \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ t_1 - t & 1 \end{bmatrix} \begin{bmatrix} 1 & -\frac{t_1-t_0}{2} \\ -\frac{t_1-t_0}{2} & \frac{(t_1-t_0)^2}{3} \end{bmatrix} \\ &= \tfrac{12p}{(t_1-t_0)^3} \begin{bmatrix} t_1 - t & 1 \end{bmatrix} \begin{bmatrix} 1 & -\frac{t_1-t_0}{2} \\ -\frac{t_1-t_0}{2} & \frac{(t_1-t_0)^2}{3} \end{bmatrix} \\ = \tfrac{12p}{(t_1-t_0)^3} &\begin{bmatrix} t_1 - t - \frac{1}{2p}(t_1 - t_0) & -\frac{1}{2p}(t_1 - t)(t_1 - t_0) + \frac{1}{3p}(t_1 - t_0)^2 \end{bmatrix} \\ &= p \begin{bmatrix} \frac{12(t_1-t)}{(t_1-t_0)^3} - \frac{6}{(t_1-t_0)^2} & -\frac{6(t_1-t)}{(t_1-t_0)^2} + \frac{4}{t_1-t_0} \end{bmatrix}. \end{aligned} \tag{4.228}$$

Substituting into (4.200) gives the optimal control

$$u(t) = \begin{bmatrix} \frac{12(t_1-t)}{(t_1-t_0)^3} - \frac{6}{(t_1-t_0)^2} & -\frac{6(t_1-t)}{(t_1-t_0)^2} + \frac{4}{t_1-t_0} \end{bmatrix} (x(t_1) - \begin{bmatrix} 1 & t_1 - t_0 \\ 0 & 1 \end{bmatrix} x(t_0)) \tag{4.229}$$

Note that this expression with $t_0 = 0$ *reduces to*

$$u(t) = \begin{bmatrix} \frac{6t_1-12t}{t_1^3} & \frac{-2t_1+6t}{t_1^2} \end{bmatrix} (x(t_1) - \begin{bmatrix} 1 & t_1 \\ 0 & 1 \end{bmatrix} x(t_0)). \tag{4.230}$$

Note that the optimal control is independent of the control weighting p*.*

Example 4.7 *Consider a system (A, B) where $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times r}$, and where A is nilpotent so that $A^2 = 0$. The transition matrix is in this case given by*

$$e^{At} = I + At. \tag{4.231}$$

See e.g. example 4.6 for a system matrix which has this property. The controllability gramian is given by

$$\begin{aligned} W_c(t_0, t) &= \int_0^{t-t_0} e^{A\tau} BP^{-1}B^T e^{A^T\tau} d\tau \\ &= \int_0^{t-t_0} (I + A\tau)BP^{-1}((I + A\tau)B)^T d\tau. \end{aligned} \tag{4.232}$$

Putting $P = I$ gives

$$W_c(t_0, t) = \begin{bmatrix} B & AB \end{bmatrix} \int_0^{t-t_0} \begin{bmatrix} I_r \\ \tau I_r \end{bmatrix} \begin{bmatrix} I_r & \tau I_r \end{bmatrix} d\tau \begin{bmatrix} B & AB \end{bmatrix}^T, \tag{4.233}$$

where I_r is the $r \times r$ identity matrix. This gives

$$W_c(t_0, t) = C_2 F(t - t_0) C_2^T \tag{4.234}$$

where

$$C_2 = \begin{bmatrix} B & AB \end{bmatrix}, \tag{4.235}$$

and

$$F(t - t_0) = \begin{bmatrix} I_r & \frac{1}{2}(t - t_0)^2 I_r \\ \frac{1}{2}(t - t_0)^2 I_r & \frac{1}{3}(t - t_0)^3 I_r \end{bmatrix}. \tag{4.236}$$

Note that this $F(t - t_0)$ matrix with $t - t_0 = 1$ is known as a Hilbert matrix which is a famous example of an ill-conditioned matrix.

Example 4.8 *Consider a system matrices $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times r}$. Assume that A is nilpotent so that $A^3 = 0$. In this case*

$$e^{At} = I + At + \frac{1}{2}A^3. \tag{4.237}$$

The controllability gramian $W_c(0, t)$ can in this case be expressed in terms of the controllability matrix as

$$W_c(0, t) = C_3 F(t) C_3^T \tag{4.238}$$

where

$$C_3 = \begin{bmatrix} B & AB & A^2B \end{bmatrix}, \tag{4.239}$$

is the controllability matrix and

$$F(t) = \begin{bmatrix} I_r & \frac{1}{2}t^2 I_r & \frac{1}{6}t^3 I_r \\ \frac{1}{2}t^2 I_r & \frac{1}{3}t^3 I_r & \frac{1}{8}t^4 I_r \\ \frac{1}{6}t^3 I_r & \frac{1}{8}t^3 I_r & \frac{1}{20}t^5 I_r \end{bmatrix}, \tag{4.240}$$

where I_r is the $r \times r$ identity matrix. It can be shown that $F(t)$ is symmetric and positive definit for all $t > 0$.

4.11 Exercises

Exercise 4.1 *Consider the system*

$$A = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{4.241}$$

and

$$B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{4.242}$$

a) Show that $A^2 = 0$.

b) Find the controllability gramian $W_c(0, 1)$.

4.12 Analytical solution to the scalar LQ problem

We will in this section study the LQ problem of a scalar system analytically. Consider the system

$$\dot{x} = ax + bu, \;\; x(t_0) \text{ given.} \tag{4.243}$$

and the performance index

$$J = \frac{1}{2}sx(t_1)^2 + \frac{1}{2}\int_{t_0}^{t_1}(qx^2 + pu^2)dt. \tag{4.244}$$

The solution to this problem is given by

$$u = -\frac{b}{p}r(t)x \tag{4.245}$$

where $r(t)$ is the positive solution to the scalar Riccati differential equation

$$-\dot{r} = 2ar - \frac{b^2}{p}r^2 + q, \;\; r(t_1) = s. \tag{4.246}$$

This differential equation can be solved analytically, e.g. by the method which is known as separation of variables.

The solution can also be derived from an eigenvalue-eigenvector decomposition of the Hamiltonian matrix. The Hamiltonian matrix F corresponding to the state and costate system

$$\begin{bmatrix} \dot{x} \\ \dot{p} \end{bmatrix} = F \begin{bmatrix} x(t) \\ p(t) \end{bmatrix} \tag{4.247}$$

is given by

$$F = \begin{bmatrix} a & -\frac{b^2}{p} \\ -q & -a \end{bmatrix}. \tag{4.248}$$

The solution is given as

$$\begin{bmatrix} x(t_1) \\ p(t_1) \end{bmatrix} = \Phi \begin{bmatrix} x(t) \\ p(t) \end{bmatrix} \tag{4.249}$$

where $\Phi = e^{F(t_1-t)}$ is the transition matrix. The transition matrix can be computed from an eigenvalue and eigenvector decomposition of F.

The two eigenvalues of matrix F, λ_1 and λ_2 are given by $\lambda_1 = -\lambda$ and $\lambda_2 = \lambda$ where

$$\lambda = \sqrt{a^2 + \frac{q}{p}b^2}. \tag{4.250}$$

Define the eigenvalue matrix as

$$\Lambda = \begin{bmatrix} -\lambda & 0 \\ 0 & \lambda \end{bmatrix}. \tag{4.251}$$

The corresponding eigenvector matrix is given by

$$M = \begin{bmatrix} 1 & 1 \\ -\frac{q}{a-\lambda} & -\frac{q}{a+\lambda} \end{bmatrix}, \tag{4.252}$$

and the inverse is

$$M^{-1} = \frac{a^2-\lambda^2}{2q\lambda} \begin{bmatrix} -\frac{q}{a+\lambda} & -1 \\ \frac{q}{a-\lambda} & 1 \end{bmatrix} = M^{-1} = \frac{b^2}{2p\lambda} \begin{bmatrix} -\frac{q}{a+\lambda} & -1 \\ \frac{q}{a-\lambda} & 1 \end{bmatrix}. \tag{4.253}$$

The transition matrix corresponding to the solution of the state and costate equations is now given by

$$\Phi(t_1 - t) = e^{F(t_1-t)} = M^{-1} e^{\Lambda(t_1-t)} M, \tag{4.254}$$

which gives

$$\Phi = \frac{a^2-\lambda^2}{2q\lambda} \begin{bmatrix} -\frac{q}{a+\lambda} e^{-\lambda(t_1-t)} + \frac{q}{a-\lambda} e^{\lambda(t_1-t)} & -e^{-\lambda(t_1-t)} + e^{\lambda(t_1-t)} \\ \frac{q^2}{a^2-\lambda^2} e^{-\lambda(t_1-t)} - \frac{q^2}{a^2-\lambda^2} e^{\lambda(t_1-t)} & \frac{q}{a-\lambda} e^{-\lambda(t_1-t)} - \frac{q}{a+\lambda} e^{\lambda(t_1-t)} \end{bmatrix} \tag{4.255}$$

and

$$\Phi = \frac{1}{2\lambda} \begin{bmatrix} -(a-\lambda)e^{-\lambda(t_1-t)} + (a+\lambda)e^{\lambda(t_1-t)} & -\frac{a^2-\lambda^2}{q} e^{-\lambda(t_1-t)} + \frac{a^2-\lambda^2}{q} e^{\lambda(t_1-t)} \\ q(e^{-\lambda(t_1-t)} - e^{\lambda(t_1-t)}) & (a+\lambda)e^{-\lambda(t_1-t)} - (a-\lambda)e^{\lambda(t_1-t)} \end{bmatrix} \tag{4.256}$$

The elements in Φ can be written in terms of the hyperbolic sine and cosine as follows

$$\begin{aligned} \phi_{11} &= \frac{1}{2\lambda}(a(e^{\lambda(t_1-t)} - e^{-\lambda(t_1-t)}) + \lambda(e^{\lambda(t_1-t)} + e^{-\lambda(t_1-t)})) \\ &= \frac{1}{\lambda}(a\sinh(\lambda(t_1-t) + \lambda\cosh(\lambda(t_1-t)), \end{aligned} \tag{4.257}$$

$$\phi_{21} = -\frac{1}{\lambda} q \sinh(\lambda(t_1-t)), \tag{4.258}$$

$$\phi_{12} = -\frac{\lambda^2-a^2}{\lambda q} \sinh(\lambda(t_1-t)), \tag{4.259}$$

$$\begin{aligned} \phi_{22} &= \frac{1}{2\lambda}(-a(e^{\lambda(t_1-t)} - e^{-\lambda(t_1-t)}) + \lambda(e^{\lambda(t_1-t)} + e^{-\lambda(t_1-t)})) \\ &= \frac{1}{\lambda}(-a\sinh(\lambda(t_1-t) + \lambda\cosh(\lambda(t_1-t)). \end{aligned} \tag{4.260}$$

The solution to the scalar Riccati equation is then

$$r(t) = \frac{s\phi_{11} - \phi_{21}}{\phi_{22} - s\phi_{12}}. \tag{4.261}$$

This gives

$$r(t) = \frac{q\tanh(\lambda(t_1-t)) + s(a\tanh(\lambda(t_1-t)) + \lambda)}{\lambda - a\tanh(\lambda(t_1-t)) + s\frac{\lambda^2-a^2}{q}\tanh(\lambda(t_1-t))}. \quad (4.262)$$

Assume $s=0$. Then

$$r(t) = \frac{q\sinh(\lambda(t_1-t))}{\lambda\cosh(\lambda(t_1-t)) - a\sinh(\lambda(t_1-t))} = \frac{q\tanh(\lambda(t_1-t)}{\lambda - a\tanh(\lambda(t_1-t))}. \quad (4.263)$$

Note that the hyperbolic cosine and sine of a number z are defined as $\sinh(z) = \frac{1}{2}(e^z - e^{-z})$ and $\cosh(z) = \frac{1}{2}(e^z + e^{-z})$. The hyperbolic tangent is defined as $\tanh(z) = \frac{\sinh(z)}{\cosh(z)}$ and the hyperbolic cotangent is defined as $\coth(z) = \frac{1}{\tanh(z)}$.

Assume $t_1 \to \infty$. This gives the scalar algebraic Riccati equation and the positive solution is found from the above as

$$r_\infty = \lim_{t_1\to\infty} r(t) = \frac{q + s(\lambda+a)}{\lambda - a + \frac{s(\lambda^2-a^2)}{q}} = \frac{q}{\lambda-a}\frac{1+\frac{s(\lambda+a)}{q}}{1+\frac{s(\lambda+a)}{q}} = \frac{q}{\lambda-a}, \quad (4.264)$$

which is independent of the final state weighting s. We have here used that

$$\tanh(z) = \frac{e^z - e^{-z}}{e^z + e^{-z}} = \frac{1-e^{-2z}}{1+e^{-2z}} \quad (4.265)$$

and, with $z = \lambda(t_1 - t)$ that

$$\lim_{t_1\to\infty} \tanh(\lambda(t_1-t)) = \lim_{t_1\to\infty} \frac{1-e^{-2\lambda(t_1-t)}}{1+e^{-2\lambda(t_1-t)}} = 1. \quad (4.266)$$

Note that an alternative expression for r_∞ is found by solving for the positive solution to the ARE, i.e. solving for $r > 0$ where $-\frac{b^2}{p}r^2 + 2ar + q = 0$, which gives $r_\infty = \frac{p}{b^2}(a+\lambda)$.

Let us study the relationship between the weights and the closed loop system in this case. We now that $-\lambda$ is the eigenvalue of the closed loop system Hence, from $-\lambda = a - bg$ where $g = \frac{b}{p}r_\infty$ we obtain

$$\frac{q}{p} = \frac{\lambda^2 - a^2}{b^2}. \quad (4.267)$$

This means that it is possible to specify the closed loop eigenvalue $-\lambda$ and compute the corresponding ratio between the weights. This result is generalized to general linear systems in Solheim (1972) (eigenvector-eigenvalue method) Di Ruscio (1990) (Schur method).

4.12.1 The case with $q = 0$ in the objective function

Consider the case with no intermediate state weighting, i.e., $q = 0$ in the objective function. The solution to the Riccati equation is in this case given by

$$\begin{aligned} r(t) &= \frac{s(a\tanh(a(t_1-t)) + a)}{a - a\tanh(a(t_1-t)) + s\frac{b^2}{p}\tanh(a(t_1-t))} \\ &= \frac{s}{\frac{sb^2}{2ap} + (1 - \frac{sb^2}{2ap})e^{-2a(t_1-t)}} \end{aligned} \quad (4.268)$$

Consider now an infinite horizon LQ problem with zero state weighting.

Unstable system $a > 0$

The steady state value of $r(t)$ as $t_1 - t \to \infty$ is in this case given by

$$r_\infty = \frac{2ap}{b^2}. \tag{4.269}$$

The closed loop system is in this case $\dot{x} = a_{cl}x$ where

$$a_{cl} = a - \frac{b^2}{p} r_\infty = -a. \tag{4.270}$$

This means e.g. that the LQ optimal feedback with zero state weighting, i.e. $g = -\frac{b}{p} r_\infty = -2a$, will stabilize an unstable system.

The algebraic Riccati equation is in the case with $Q = 0$ reduced to a Lyapunov equation in R^{-1}, i.e. $R^{-1}A^T + AR^{-1} - BP^{-1}B^T = 0$. In the scalar case with $q = 0$ we get $r^{-1} = \frac{b^2}{2ap}$ and $r = \frac{2ap}{b^2}$.

The result above can be generalized to multivariable linear systems and is known in the literature as the *mirror image property* of the LQ regulator. It states that the eigenvalues of a closed loop LQ system, obtained with zero state weighting $Q = 0$, is identical to $-\lambda(A)$ where $\lambda(A)$ is the open loop eigenvalues.

Stable open loop system $a < 0$

The steady state value of $r(t)$ as $t_1 - t \to \infty$ is in this case

$$r_\infty = 0 \tag{4.271}$$

and the closed loop system is $\dot{x} = ax$ and $u = 0$ is the optimal control.

Integrator $a = 0$

The case where both $q = 0$ and $a = 0$ needs to be handled separately. In this case we have that the transition matrix of the state and costate system is

$$\Phi(t_1 - t) = e^{F(t_1 - t)} = I + F(t_1 - t) = \begin{bmatrix} 1 & -\frac{b^2}{p}(t_1 - t) \\ 0 & 1 \end{bmatrix}. \tag{4.272}$$

The solution to the Riccati equation is then

$$r(t) = \frac{s\phi_{11}}{\phi_{22} - s\phi_{12}} = \frac{s}{1 + \frac{sb^2}{p}(t_1 - t)}. \tag{4.273}$$

Consider $t_1 \to \infty$. Then we have that $r_\infty = 0$ and that $u = 0$ is the optimal control.

Example 4.9 (Temperature control in a room)
Define $\theta(t)$ as the temperature in the room, θ_a as the ambient temperature (Norwegian: omgivelses temperatur) which is assumed to be constant, and $u(t)$ as the rate of heat supply to the room. The dynamics of the room temperature is then given by

$$\dot{\theta} = -\lambda(\theta - \theta_a) + bu, \tag{4.274}$$

where λ and b are constants.

Define the state as

$$x(t) = \theta - \theta_d, \tag{4.275}$$

where θ_d is the desired room temperature. Then we have the state equation

$$\dot{x} = ax + bu + v, \tag{4.276}$$

where $a = -\lambda$ and $v = a(\theta_a - \theta_d)$.

Consider the following objective function

$$\begin{aligned} J &= \frac{1}{2}s(\theta(t_1) - \theta_d)^2 + \frac{1}{2}\int_{t_0}^{t_1}(q(\theta - \theta_d)^2 + pu^2)dt \\ &= \frac{1}{2}sx(t_1)^2 + \frac{1}{2}\int_{t_0}^{t_1}(qx^2 + pu^2)dt. \end{aligned} \tag{4.277}$$

A special case of interest is to put $q = 0$. This means that we want the temperature to be close to the desired temperature at the final time t_1 while using the least possible supplied energy.

In order to solve this problem properly we need a method to incorporate external signals in the model, i.e., the disturbance in the state equation. This will be discussed later.

However, the above problem can be re-formulated by defining the state as

$$x(t) = \theta - \theta_a. \tag{4.278}$$

This gives the model

$$\dot{x} = ax + bu, \tag{4.279}$$

and the objective (with $q = 0$)

$$J = \frac{1}{2}s(x(t_1) - x_r)^2 + \frac{1}{2}\int_{t_0}^{t_1} pu^2 dt. \tag{4.280}$$

where

$$x_r = \theta_d - \theta_a \tag{4.281}$$

can be viewed as a reference signal for the final state. x_r is assumed to be known for all times $t_0 \leq t \leq t_1$.

Let us study this problem in detail. The solution of the state and costate system is

$$\begin{bmatrix} x(t_1) \\ p(t_1) \end{bmatrix} = \begin{bmatrix} \phi_{11} & \phi_{12} \\ 0 & \phi_{22} \end{bmatrix} \begin{bmatrix} x(t) \\ p(t) \end{bmatrix} \tag{4.282}$$

where

$$\phi_{11} = e^{a(t_1-t)}, \tag{4.283}$$

$$\phi_{22} = e^{-a(t_1-t)}, \tag{4.284}$$

$$\phi_{12} = -\frac{b^2}{ap}\sinh(a(t_1-t)). \tag{4.285}$$

Note that the transition matrix Φ of an upper triangular matrix F has the same structure as F and that F and Φ commutes, i.e. $F\Phi = \Phi F$. Note also that the diagonal elements in Φ is equal to the exponent of the corresponding diagonal elements in F.

The optimal control is given by

$$u(t) = -\frac{b}{p}p(t). \tag{4.286}$$

We will no go for a relationship between $p(t)$ and $x(t)$.

The boundary condition $p(t_1)$ is found from the maximum principle, i.e.,

$$p(t_1) = \frac{\partial}{\partial t_1}\frac{1}{2}s(x(t_1)-x_r)^2 = s(x(t_1)-x_r). \tag{4.287}$$

Hence, we have three equations

$$x(t_1) = \phi_{11}x + \phi_{12}p, \tag{4.288}$$

$$p(t_1) = \phi_{22}p, \tag{4.289}$$

$$p(t_1) = s(x(t_1)-x_r), \tag{4.290}$$

which gives

$$p = r(t)x + h(t), \tag{4.291}$$

where

$$r(t) = \frac{s\phi_{11}}{\phi_{22} - s\phi_{12}}, \tag{4.292}$$

and

$$h(t) = -\frac{s}{\phi_{22} - s\phi_{12}}x_r. \tag{4.293}$$

The optimal control is then given by

$$u(t) = g_1(t)x(t) + g_2(t)x_r, \tag{4.294}$$

where

$$g_1(t) = -\frac{b}{p}r(t), \tag{4.295}$$

$$g_2(t) = \frac{b}{p}\frac{s}{\phi_{22} - s\phi_{12}}. \tag{4.296}$$

Hence, the optimal control consist of a feedback from the state and a feedforward from the reference.

An expression for the final state $x(t_1)$ can be found as follows. Using (4.288), (4.289) and (4.290) with $t = t_0$ gives the final state $x(t_1)$ as a function of known variables, i.e.,

$$x(t_1) = \frac{\phi_{11}(t_0,t_1)}{1+sW_c(t_0,t_1)}x_0 + \frac{sW_c(t_0,t_1)}{1+sW_c(t_0,t_1)}x_r \tag{4.297}$$

where

$$W_c(t_0,t_1) = -\frac{\phi_{12}}{\phi_{22}} = \frac{b^2}{ap}e^{a(t_1-t_0)}\sinh(a(t_1-t_0)) \tag{4.298}$$

is the weighted controllability gramian for the pair (a,b) and weight p. Note that $x(t_1) = x_r$ as $s \to \infty$.

The results of the LQ optimal control strategy are illustrated in Figures 4.4 and 4.5.

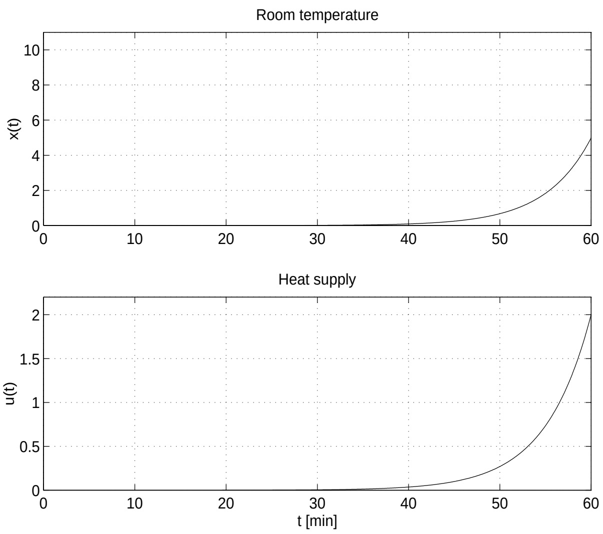

Figure 4.4: Optimal control of room temperature. $s = 0.4$, $p = 1$, $x_r = 10$, $a = -1/5$, $b = 1$, $t_0 = 0$, $t_1 = 60$, $x(t_0) = 0$. An LQ optimal control $u(t)$ is used for $t_0 \le t \le t_1$.

Suppose that the heat supply $u(t)$ is held constant equal to $u(t_1)$ for all times $t > t_1$ and that we want to find the weight s such that

$$\lim_{t\to\infty} x(t) = x_r. \tag{4.299}$$

The steady state control is in this case $u_s = -\frac{a}{b}x_r$ Hence, an equation for the weight is determined from $u(t_1) = u_s$. We have

$$u(t_1) = -\frac{b}{p}p(t_1) \tag{4.300}$$

where

$$p(t_1) = s(x(t_1) - x_r) = \frac{s\phi_{11}(t_0,t_1)}{1+sW_c(t_0,t_1)}x_0 - \frac{s}{1+sW_c(t_0,t_1)}x_r \tag{4.301}$$

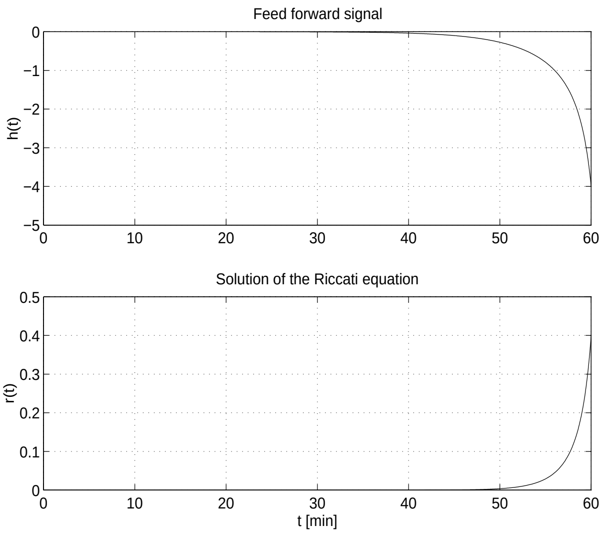

Figure 4.5: Optimal control of room temperature. $s = 0.4$, $p = 1$, $x_r = 10$, $a = -1/5$, $b = 1$, $t_0 = 0$, $t_1 = 60$. An LQ optimal control $u(t)$ is used for $t_0 \leq t \leq t_1$.

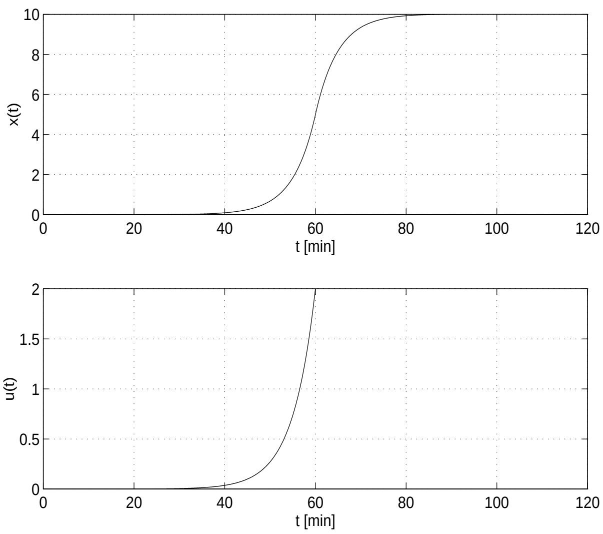

Figure 4.6: Optimal control of room temperature. $s = 0.4$, $p = 1$, $x_r = 10$, $a = -1/5$, $b = 1$, $t_0 = 0$, $t_1 = 60$, $x(t_0) = 0$. An LQ optimal control $u(t)$ is used for $t_0 \leq t \leq t_1$. The control is held constant $u(t) = u(t_1)$ for $t > t_1$.

Putting (4.301) into (4.300) and solving for s gives

$$s = \frac{u(t_1)}{\frac{b}{p}x_r - \frac{b}{p}\phi_{11}(t_0, t_1)x_0 - W_c(t_0, t_1)u(t_1)}. \tag{4.302}$$

This control strategy is simulated and illustrated in Figure 4.6.

Example 4.10 (Temperature control in a room)
Consider the same problem as in Example 4.9 but with parameters $t_0 = t$ and $t_1 =$

$t+T$ where T is a constant time horizon. Hence, we have a receding horizon objective

$$J = \frac{1}{2}s(x(t+T)-x_r)^2 + \frac{1}{2}\int_t^{t+T} pu^2 dt. \tag{4.303}$$

The solution to this control problem is found by putting $t_1 = t+T$ into the control determined in Example 4.9. We have

$$u(t) = g_1(T)x(t) + g_2(T)x_r \tag{4.304}$$

where $g_1(T)$ and $g_2(T)$ now is constant parameters defined as follows

$$g_1(T) = -\frac{b}{p}r(T) = -\frac{b}{p}\frac{s\phi_{11}(T)}{\phi_{22}(T)-s\phi_{12}(T)}, \tag{4.305}$$

$$g_2(T) = \frac{b}{p}\frac{s}{\phi_{22}(T)-s\phi_{12}(T)}. \tag{4.306}$$

where

$$\phi_{11}(T) = e^{aT}, \tag{4.307}$$
$$\phi_{22}(T) = e^{-aT}, \tag{4.308}$$
$$\phi_{12}(T) = -\frac{b^2}{ap}\sinh(aT). \tag{4.309}$$

The closed loop system with this control is given by

$$\dot{x} = (a+bg_1(T))x + bg_2(T)x_r. \tag{4.310}$$

Define the closed loop pole as

$$a_{cl} = a + bg_1(T) \tag{4.311}$$

and the steady state as

$$x_s = \lim_{t\to\infty} x(t) = \frac{-bg_2(T)}{a+bg_1(T)}x_r. \tag{4.312}$$

The steady state x_s and the closed loop pole $a_{cl} = a + bg_1(T)$ are illustrated as a function of the weight s and the horizon T in Figures 4.7 and 4.8.

The figures shows that a small horizon T and a large weight s will result in a steady state x_s which is close to the reference x_r. Note however, that the closed loop system is very fast with T small and s large. Note also that there are no finite parameters $s > 0$ and $T > 0$ which will result in a steady state x_s which is identically equal to x_r.

The heat supply $u(t)$ and the state $x(t)$, (which is defined as the difference between the room temperature and the ambient temperature), are illustrated in Figure 4.9. Note that the heat supply at time $t = 0$ is different from zero in this case. The control at time zero is $u(0) = g_2(T)x(0) + g_2(T)x_r = g_2(T)x_r$ in this case. Note that the control was $u(t = 0) = 0$ for the LQ optimal control strategy in Example 4.9, Figures 4.4 and 4.6.

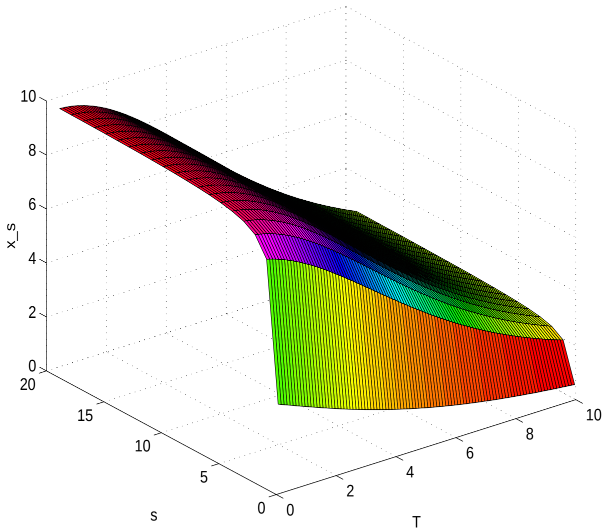

Figure 4.7: Steady state temperature with predictive control, as a function of the weight s and the horizon T. $a = -1/5$, $b = 1$, $p = 1$, $x_r = 10$.

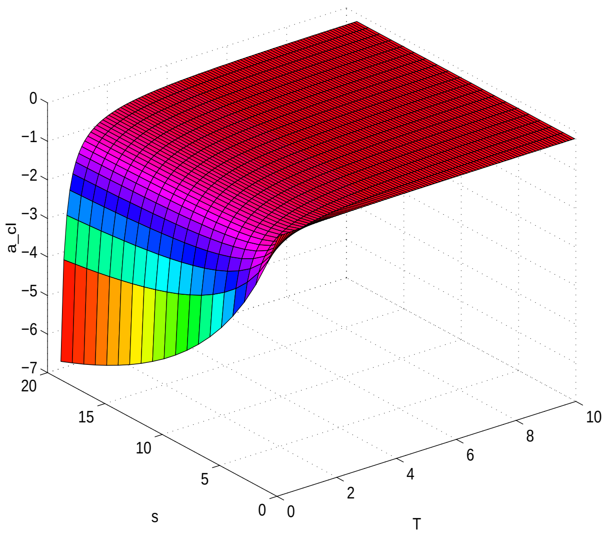

Figure 4.8: The closed loop pole $a_{cl} = a + bg_1(T)$ as a function of the weight s and the horizon T. $a = -1/5$, $b = 1$, $p = 1$, $x_r = 10$.

4.13 Analytical solution to the tracking problem

Consider a linear model $\dot{x} = Ax + Bu + Cr$, $y = Dx$, initial values $x(t_0)$ specified and the performance index

$$J = \frac{1}{2}(r(t_1) - y(t_1))^T S(r(t_1) - y(t_1)) + \frac{1}{2}\int_{t_0}^{t_1} (r - y)^T Q(r - y) + u^T Pu)dt \tag{4.313}$$

We will in the following discuss the solution to the optimal tracking problem. An analytical derivation will be given as far as possible.

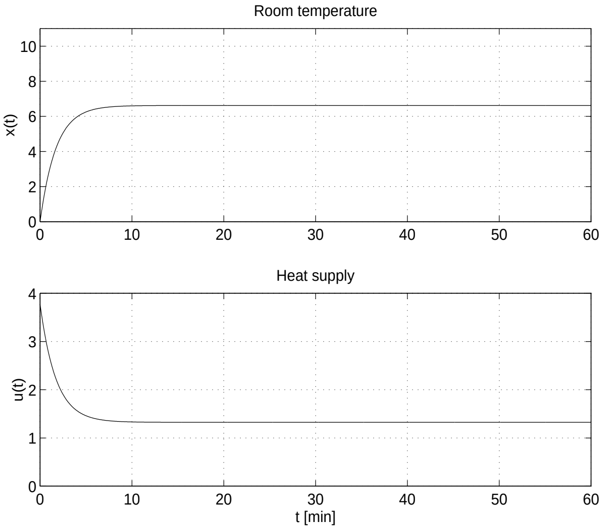

Figure 4.9: The state and heat supply with predictive control. $a = -1/5$, $b = 1$, $p = 1$, $s = 0.4$, $x_r = 10$, $T = 0.1$, $x(t = 0) = 0$.

The Hamiltonian function is

$$H = \frac{1}{2}[(r - Dx)^T Q(r - Dx) + u^T Pu)] + p^T(Ax + Bu + Cr) \tag{4.314}$$

The optimal control is determined from the 1st order condition for a minimum, i.e. $\frac{\partial H}{\partial u} = 0$, which gives $u = -P^{-1}B^T p(t)$. We will in the following prove the relationship $p = Rx + h$.

Derivation of the relationship $p = Rx + h$

The co-state is given by $\dot{p} = -\frac{\partial H}{\partial x}$. Having that

$$\frac{\partial H}{\partial x} = -D^T Q(r - Dx) + A^T p \tag{4.315}$$

gives

$$\dot{p} = -(-D^T Q(r - Dx) + A^T p) = -D^T QDx - A^T p + D^T Qr. \tag{4.316}$$

Note that the derivative of a vector valued scalar function $f(u(x))$ with respect to a vector x is given by $\frac{\partial f}{\partial x} = (\frac{\partial u}{\partial x})^T \frac{\partial f}{\partial u}$. This can be used to find the derivative of the first term in the Hamilton function. Consider the quadratic function $f = \frac{1}{2}(r - Dx)^T Q(r - Dx)$. Defining $u = r - Dx$ gives $f = \frac{1}{2}u^T Qu$, $\frac{\partial u}{\partial x} = -D$, $\frac{\partial f}{\partial u} = Qu$ and $\frac{\partial f}{\partial x} = -D^T Qu$.

The state equation substituted for the optimal control is

$$\dot{x} = Ax - BP^{-1}B^T p + Cr. \tag{4.317}$$

This gives the system of differential equations

$$\begin{bmatrix} \dot{x} \\ \dot{p} \end{bmatrix} = \overbrace{\begin{bmatrix} A & -BP^{-1}B^T \\ -D^T QD & -A^T \end{bmatrix}}^{F} \begin{bmatrix} x \\ p \end{bmatrix} + \begin{bmatrix} C \\ D^T Q \end{bmatrix} r \tag{4.318}$$

The solution is

$$\begin{bmatrix} x(t_1) \\ p(t_1) \end{bmatrix} = \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{bmatrix} \begin{bmatrix} x \\ p \end{bmatrix} + \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} \tag{4.319}$$

where we have defined

$$\Phi(t_1 - t) = \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{bmatrix} = e^{F(t_1-t)} \tag{4.320}$$

and

$$\begin{bmatrix} h_1 \\ h_2 \end{bmatrix} = \int_t^{t_1} e^{F(t_1-\tau)} \begin{bmatrix} C \\ D^T Q \end{bmatrix} r(\tau) d\tau \tag{4.321}$$

The transition matrix (4.320) and the integral (4.321) can for some simple systems be solved analytically. The transition matrix can also be defined via the eigenvalue decomposition or the Schur form of matrix F.

The boundary condition for (4.316) is given by

$$p(t_1) = \frac{\partial H}{\partial x(t_1)} [\frac{1}{2}(r(t_1) - Dx(t_1))^T S(r(t_1) - Dx(t_1))] = -D^T S(r(t_1) - Dx(t_1))$$

Write this for convenience with the literature as

$$p(t_1) = R(t_1)x(t_1) + h(t_1), \tag{4.322}$$

where

$$R(t_1) = D^T SD, \quad h(t_1) = -D^T Sr(t_1). \tag{4.323}$$

The point is now that we have three equations (4.319) and (4.322), which can be combined to give

$$p = R(t)x + h(t), \tag{4.324}$$

where we have defined

$$R(t) = (\Phi_{22} - R(t_1)\Phi_{21})^{-1}(R(t_1)\Phi_{11} - \Phi_{21}) \tag{4.325}$$

$$h(t) = (\Phi_{22} - R(t_1)\Phi_{21})^{-1}(R(t_1)h_1 - h_2 + h(t_1)) \tag{4.326}$$

Note that $R(t)$ is the solution to the Riccati equation $-\dot{R} = A^T R + RA - RBP^{-1}B^T R + D^T QD$ with final value $R(t_1) = D^T SD$, and that $h(t)$ is a solution to the differential equation $-\dot{h} = (A - BP^{-1}B^T R(t))^T h + (RC - D^T Q)r$ with final value as above, i.e. $h(t_1) = -D^T Sr(t_1)$. $h(t)$ is often refereed to as a feed-forward signal.

Tracking a step change

Define the reference as

$$r(t) = \begin{cases} 0 & \forall\ t_0 \leq t < t_s \\ r_0 & \forall\ t_s \leq t \leq t_1 \end{cases} \tag{4.327}$$

where r_0 is a constant.

In order to compute the feed-forward signal from (4.326) we have to define the signals h_1 and h_2.

For $t_0 \le t < t_s$ we have

$$\begin{aligned}\begin{bmatrix} h_1 \\ h_2 \end{bmatrix} &= \int_t^{t_1} e^{F(t_1-\tau)} \begin{bmatrix} C \\ D^T Q \end{bmatrix} r(\tau) d\tau \\ &= \int_t^{t_s} e^{F(t_1-\tau)} \begin{bmatrix} C \\ D^T Q \end{bmatrix} \overbrace{r(\tau)}^{=0} d\tau + \int_{t_s}^{t_1} e^{F(t_1-\tau)} \begin{bmatrix} C \\ D^T Q \end{bmatrix} \overbrace{r(\tau)}^{=r_0} d\tau \\ &= (\int_{t_s}^{t_1} e^{F(t_1-\tau)} d\tau) \begin{bmatrix} C \\ D^T Q \end{bmatrix} r_0 \end{aligned} \tag{4.328}$$

Hence, the problem is a function of the transition matrix. Note that if F is non-singular

$$\begin{bmatrix} h_1 \\ h_2 \end{bmatrix} = F^{-1}(e^{F(t_1-t_s)} - I_{2n}) \begin{bmatrix} C \\ D^T Q \end{bmatrix} r_0 \tag{4.329}$$

Remark that h_1 and h_2 are constant vectors in this case.

For $t_s \le t < t_1$ we have

$$\begin{aligned}\begin{bmatrix} h_1 \\ h_2 \end{bmatrix} &= \int_t^{t_1} e^{F(t_1-\tau)} \begin{bmatrix} C \\ D^T Q \end{bmatrix} r(\tau) d\tau \\ &= (\int_t^{t_1} e^{F(t_1-\tau)} d\tau) \begin{bmatrix} C \\ D^T Q \end{bmatrix} r_0 \end{aligned} \tag{4.330}$$

and for F non-singular

$$\begin{bmatrix} h_1 \\ h_2 \end{bmatrix} = F^{-1}(e^{F(t_1-t)} - I_{2n}) \begin{bmatrix} C \\ D^T Q \end{bmatrix} r_0 \tag{4.331}$$

Remark that h_1 and h_2 are in general time variant functions in this case. The problem of computing h_1 and h_2 is relatively simple in case of a constant reference or a step change in the reference signal. Constant step change reference signals is also frequently used in practice.

Example 4.11
For a scalar system

$$\dot{x} = ax + bu \tag{4.332}$$

$$y = x \tag{4.333}$$

where the initial state $x(t_0)$ is given and with performance index

$$J = \frac{1}{2} s(r(t_1) - y(t_1))^2 + \frac{1}{2} \int_{t_0}^{t_1} q(r-y)^2 + pu^T dt. \tag{4.334}$$

we have that the elements in the transition matrix $\Phi = e^{F(t_1-t)}$ *are given by*

$$\begin{aligned}
\phi_{11} &= \frac{a}{\lambda}\sinh(\lambda(t_1-t)) + \cosh(\lambda(t_1-t)), && (4.335)\\
\phi_{21} &= -\frac{q}{\lambda}\sinh(\lambda(t_1-t)), && (4.336)\\
\phi_{12} &= -\frac{\lambda^2-a^2}{\lambda q}\sinh(\lambda(t_1-t)), && (4.337)\\
\phi_{22} &= -\frac{a}{\lambda}\sinh(\lambda(t_1-t)) + \cosh(\lambda(t_1-t)). && (4.338)
\end{aligned}$$

The solution is

$$u = -\frac{b}{P}p(t) \tag{4.339}$$

where the co-state is

$$p = r(t)x + h(t) \tag{4.340}$$

where

$$\begin{aligned}
r(t) &= \frac{s\phi_{11}-\phi_{21}}{\phi_{22}-s\phi_{12}} && (4.341)\\
h(t) &= \frac{sh_1-h_2-sr(t_1)}{\phi_{22}-s\phi_{12}} && (4.342)
\end{aligned}$$

In order to compute $h(t)$ *we need to find* h_1 *and* h_2*. The reference is a step change from zero to* r_0 *at time* t_s*, i.e. as defined in (4.327). We can use (4.329) and (4.330) directly.*

For $t_0 \le t < t_s$ *we have*

$$\begin{aligned}
\begin{bmatrix} h_1 \\ h_2 \end{bmatrix} &= \int_{t_s}^{t_1} e^{F(t_1-\tau)}d\tau \begin{bmatrix} 0 \\ q \end{bmatrix} r_0 = \int_{t_s}^{t_1} \begin{bmatrix} \phi_{12} \\ \phi_{22} \end{bmatrix} d\tau q r_0 \\
&= \begin{bmatrix} \frac{\lambda^2-a^2}{\lambda^2}\cosh(\lambda(t_1-\tau)) \\ \frac{qa}{\lambda^2}\cosh(\lambda(t_1-\tau)) - \frac{q}{\lambda}\sinh(\lambda(t_1-\tau)) \end{bmatrix}_{t_s}^{t_1} q r_0 && (4.343)
\end{aligned}$$

which gives

$$\begin{bmatrix} h_1 \\ h_2 \end{bmatrix} = \begin{bmatrix} \frac{\lambda^2-a^2}{\lambda^2}(1-\cosh(\lambda(t_1-t_s))) \\ \frac{qa}{\lambda^2}(1-\cosh(\lambda(t_1-t_s))) + \frac{q}{\lambda}\sinh(\lambda(t_1-t_s)) \end{bmatrix} r_0, \ \ t_0 \le t < t_s \tag{4.344}$$

and

$$\begin{bmatrix} h_1 \\ h_2 \end{bmatrix} = \begin{bmatrix} \frac{\lambda^2-a^2}{\lambda^2}(1-\cosh(\lambda(t_1-t))) \\ \frac{qa}{\lambda^2}(1-\cosh(\lambda(t_1-t))) + \frac{q}{\lambda}\sinh(\lambda(t_1-t)) \end{bmatrix} r_0, \ \ t_s \le t < t_1 \tag{4.345}$$

Chapter 5

Optimal Control of Discrete Time Systems

5.1 The discrete maximum principle

Given a discrete time dynamic process described by the model

$$x_{k+1} - x_k = f(x_k, u_k, k), \tag{5.1}$$

where k is discrete time. $f(\cdot)$ is in general a nonlinear vector function.

Furthermore, we assume an optimal performance index (criterion) of the form

$$J_i = S(x_N) + \sum_{k=i}^{N-1} L(x_k, u_k), \tag{5.2}$$

where $S(\cdot)$ is a scalar weighting function of the state at the final time instant N, $L(\cdot,\cdot)$ is a scalar weighting function of the state vector x_k and the control input vector u_k over the time horizon $i \leq k \leq N-1$. Both $S(\cdot)$ and $L(\cdot,\cdot)$ may be non linear functions.

By investigating this criterion we se that the discrete start time is $k = i$ and that the discrete final time is $k = N$. We assume that $N > i$. The criterion is defined over a time horizon of $N - i + 1$ discrete time instants. We also observe that the criterion only is dependent of the control inputs at $N - i$ time instants. Hence, this means that a part ov the criterion is not dependent of the unknown control inputs, and the criterion may be splitted into two parts. More of this later on.

We will in the following present the discrete time Maximum Principle which is a method for solving the discrete time optimal control problem

We define the discrete time Hamiltonian function corresponding to the continuous case. We have

$$\begin{aligned} H_k &= L(x_k, u_k) + p_{k+1}^T f(x_k, u_k, k) \\ &= L(x_k, u_k) + p_{k+1}^T (x_{k+1} - x_k). \end{aligned} \tag{5.3}$$

In order for the existence of an optimal control which minimize the criterion J_i it is necessary that:

- The impulse vector, p, and the state vector, x, satisfy the differential equations

$$x_{k+1} - x_k = \frac{\partial H_k}{\partial p_{k+1}} = f(x_k, u_k, k), \tag{5.4}$$

$$p_{k+1} - p_k = -\frac{\partial H_k}{\partial x_k}, \tag{5.5}$$

  with known boundary (initial and final value) conditions

$$x_i = x_0, \tag{5.6}$$

$$p_N = \frac{\partial S}{\partial x_N}. \tag{5.7}$$

  The state space model (5.1) have boundary conditions at the initial time instant. But remark that the model for the impulse vector (5.7) have boundary condition at the final time instant. This is defined as a two-point boundary value problem.

- The Hamiltonian function, H_k, must have an a absolute minimum (ore maximum) with respect to the unknown control $u_k \in U$ where U is the allowed control space. This must hold for all time instants $k = i, \cdots, N-1$. This means that we may include constraints on the control vector u_k. Those constraints define the control space U.

  Conditions for a minimum is that

$$\frac{\partial H_k}{\partial u_k} = 0, \tag{5.8}$$

  and

$$\frac{\partial^2 H_k}{\partial u_k^2} > 0. \tag{5.9}$$

5.2 Discrete optimal control of linear dynamic systems

Assume that the process may be described by the discrete time state space model

$$x_{k+1} = A_k x_k + B_k u_k, \tag{5.10}$$

where $x_k \in \mathbb{R}^n$ is the state vector of the dynamic process and $u_k \in \mathbb{R}^r$ is the control vector. $A_k \in \mathbb{R}^{n \times n}$ is the transition matrix which in general may be time variant $B_k \in \mathbb{R}^{n \times r}$ is the control input system matrix.

Consider an optimal criterion of the Linear Quadratic (LQ) form

$$J_i = \frac{1}{2} x_N^T S_N x_N + \frac{1}{2} \sum_{k=i}^{N-1} (x_k^T Q_k x_k + u_k^T P_k u_k), \tag{5.11}$$

where S_N, Q_k and P_k are symmetric weighting matrices. Note that the weighting matrices in general may be time variant. We will later on specify further detectability assumptions on the weighting matrices.

We will in the following find the optimal control, u_k^*, which minimize the optimal criterion Equation (5.11). We start by writing down the Hamiltonian function, i.e.,

$$H_k = \frac{1}{2}(x_k^T Q_k x_k + u_k^T P_k u_k) + p_{k+1}^T(\overbrace{(A_k - I)x_k + B_k u_k}^{x_{k+1}-x_k}). \tag{5.12}$$

We have used that the state space model equation (5.10) may be written as

$$x_{k+1} - x_k = (A_k - I)x_k + B_k u_k. \tag{5.13}$$

The optimal control is then found from

$$\frac{\partial H_k}{\partial u_k} = P_k u_k + B_k^T p_{k+1} = 0, \tag{5.14}$$

which gives give

$$u_k = -P_k^{-1} B_k^T p_{k+1}. \tag{5.15}$$

if the weighting matrix is non-singular (invertible). One should note that we later on, in Sec 5.2.2, will present a version which does not involve the inversion of the weighting matrix P_k.

Putting equation (5.14) into the linear state space model gives

$$x_{k+1} = A_k x_k - B_k P_k^{-1} B_k^T p_{k+1}. \tag{5.16}$$

We will later on use this expression for x_{k+1} in order for defining an expression for the optimal control. The impulse vector is defined from Equation (5.5). We have

$$p_{k+1} - p_k = -\frac{\partial H_k}{\partial x_k} = -Q_k x_k - (A_k - I)^T p_{k+1}, \tag{5.17}$$

which may be presented simply as

$$p_k = Q_k x_k + A_k^T p_{k+1}. \tag{5.18}$$

Equations (5.16) and (5.18) defines an autonomous system, i.e.,

$$\begin{bmatrix} x_{k+1} \\ p_k \end{bmatrix} = \begin{bmatrix} A_k & -H \\ Q_k & A_k^T \end{bmatrix} \begin{bmatrix} x_k \\ p_{k+1} \end{bmatrix}, \tag{5.19}$$

where the matrix H is defined as

$$H = B_k P_k^{-1} B_k^T. \tag{5.20}$$

This matrix should not be compared with the Hamiltonian function H_k.

Note that in Equation (5.19) the state vector and the impulse vector are defined at different time instants at the same side of the equality sign. In case when A_k is non-singular we find from (5.16) that

$$x_k = A_k^{-1} x_{k+1} + A_k^{-1} H p_{k+1}. \tag{5.21}$$

Putting this into (5.18) we find that

$$p_k = Q_k A_k^{-1} x_{k+1} + (A_k^T + Q_k A_k^{-1} H) p_{k+1}. \tag{5.22}$$

Equations (5.21) and (5.22) may be written in matrix form as follows

$$\begin{bmatrix} x_k \\ p_k \end{bmatrix} = \overbrace{\begin{bmatrix} A_k^{-1} & A_k^{-1} H \\ Q_k A_k^{-1} & A_k^T + Q A_k^{-1} H \end{bmatrix}}^{F} \begin{bmatrix} x_{k+1} \\ p_{k+1} \end{bmatrix}. \tag{5.23}$$

Note that the transition matrix A_k is invertible if the model is obtained by discretizing a continuous time model. You should note that (5.23) may be used in order to show that there is a linear relationship between p_k and x_k, i.e., $p_k = R_k x_k$ as well as to find an equation for R_k.

The prof of this is as follows. From (5.7) we find the boundary condition $p_N = S_N x_N$. This indicates that there is a linear relationship between x_k and p_k. Putting $k = N - 1$ in (5.23) gives, with using the boundary conditions, two equations with three unknown, p_{N-1}, x_{N-1} og x_N. Eliminating x_N we find the linear relationship

$$p_{N-1} = R_{N-1} x_{N-1}, \tag{5.24}$$

$$R_{N-1} = (F_{21} + F_{22} S_N)(F_{11} + F_{12} S_N)^{-1}. \tag{5.25}$$

Putting $k = N - 2$ into (5.23) and doing the same, i.e., finding a linear relationship between p_{N-2} and x_{N-2}. Since that we have a series to do, we use the induction principle for the prof, i.e., we can prove that there is a linear relationship between p_k and x_k. We will later on generalize this to hold also when A_k is singular.

In the same way as in the continuous case, and which is sketched above, we may show that there is a linear relationship between the impulse vector, p_k, and the state vector, x_k. Hence, we may show and assume that

$$p_k = R_k x_k. \tag{5.26}$$

This means that if we may find an equation for defining/computing R_k then we indeed have proved that there exist such a relationship as described above. This also indicates an alternative prof of the LQ optimal solution to the one given above. This prof is presented in the following

Putting (5.18) into (5.26) gives

$$R_k x_k = Q_k x_k + A_k^T p_{k+1}. \tag{5.27}$$

Expressing (5.26) at time instant $k + 1$ and putting this expression into (5.27) we find

$$R_k x_k = Q_k x_k + A_k^T R_{k+1} x_{k+1}. \tag{5.28}$$

We will now find an expression for x_{k+1} and putting this into (5.28). Putting the relationship (5.26) into (5.16) gives

$$x_{k+1} = A x_k - B_k P_k^{-1} B_k^T R_{k+1} x_{k+1}. \tag{5.29}$$

From this last equation we find an expression for for x_{k+1}

$$x_{k+1} = (I + B_k P_k^{-1} B_k^T R_{k+1})^{-1} A_k x_k. \tag{5.30}$$

Note that (5.30) have to be an expression for the closed loop system. Putting equation (5.30) into (5.28) gives

$$R_k x_k = Q_k x_k + A_k^T R_{k+1} (I + B_k P_k^{-1} B_k^T R_{k+1})^{-1} A_k x_k. \tag{5.31}$$

This equation must hold for an arbitrarily state vector $x_k \neq 0$. This gives the following matrix equation for finding R_k.

$$R_k = Q_k + A_k^T R_{k+1} (I + B_k P_k^{-1} B_k^T R_{k+1})^{-1} A_k. \tag{5.32}$$

This is one formulation of the famous Riccati equation named after Count Riccati which lived in the 1600 century. However, this formulation assumes that the control weighting matrix, P_k, is non-singular. We will later show that there exist a more general formulation of the discrete Riccati equation wich does not involve the inversion of P_k.

An alternative formulation in the case when R_{k+1} is non-singular is

$$R_k = Q_k + A_k^T (R_{k+1}^{-1} + B_k P_k^{-1} B_k^T)^{-1} A_k. \tag{5.33}$$

From (5.7) we find the boundary condition

$$p_N = S_N x_N. \tag{5.34}$$

Expressing the relationship (5.26) at $k = N$ we find that

$$p_N = R_N x_N. \tag{5.35}$$

Comparison of (5.34) and (5.35) gives the boundary condition

$$R_N = S_N, \tag{5.36}$$

which gives the boundary condition for the discrete time Riccati equation. This means that the solution R_k (at time k) may be found by iterating the Riccati equation backward in time, to the present time instant k, from the final time instant, $k = N$.

An expression for the optimal control can now be found by putting (5.26) into (5.15), i.e.,

$$u_k = -P^{-1} B^T R_{k+1} x_{k+1}. \tag{5.37}$$

Putting (5.30) into (5.37) gives

$$u_k = G_k x_k, \tag{5.38}$$
$$G_k = -P^{-1} B^T R_{k+1} (I + B P^{-1} B^T R_{k+1})^{-1} A. \tag{5.39}$$

As we see, the above solution assumes that the weighting matrix P_k is non-singular. We will in the next section propose a better solution which does not involve the inversion of P_k.

Consider now the case in which the time horizon is large, i.e., $N \to \infty$, then we have that $R_{k+1} = R_k = R$ is a constant matrix. This gives us the Discrete time Algebraic Riccati Equation (DARE). Furthermore, we may show that when choosing the weighting matrices properly then the LQ optimal solution results in a stable closed loop system. In general we have that the LQ optimal control system is stable when $N \to \infty$, under the assumptions that (A, B) is stabilizable, $(\sqrt{Q}, A)$ is detectable and P a positive definite matrix. As mentioned above, there may also in certain circumstances exist an LQ optimal solution also when P is singular.

5.2.1 Derivation of the optimal control: intuitive formulation

The solution to the discrete time LQ optimal control problem may be formulated in different ways and with different equations. In case when the transition matrix A_k is non-singular then we may find p_{k+1} from Equation (5.18), i.e.,

$$p_{k+1} = A^{-T}(p_k - Q_k x_k) = A^{-T}(R_k - Q_k)x_k, \tag{5.40}$$

where we have assumed that $p_k = R_k x_k$. Putting this into the expression for the optimal control given by Equation (5.15), we find

$$u_k = G_k x_k, \tag{5.41}$$

$$G_k = -P_k^{-1} B_k^T A_k^{-T}(R_k - Q_k). \tag{5.42}$$

This solution demands that both A_k and P_k are non-singular matrices. A_k is usually non-singular. This is in particular the case when A_k is found from discretizing a continuous time model. There may however exist cases in which A_k is singular. This is the case for systems with a static component and for systems with time delay modeled as extra "dummy" states in the system in order to take care of the time delay.

5.2.2 Derivation of the optimal control: a better formulation

We may show that there exist a formulation of the discrete LQ optimal solution which does not involve the inversion of the matrices A_k and P_k. We have from the condition for a minimum $\frac{\partial H_k}{\partial u_k} = P_k u_k + B_k^T p_{k+1} = 0$, equation (5.14), that

$$P_k u_k = -B_k^T R_{k+1} x_{k+1}, \tag{5.43}$$

where we have assumed $p_{k+1} = R_{k+1} x_{k+1}$. Putting the state space model into (5.43) gives

$$P_k u_k = -B_k^T R_{k+1}(A_k x_k + B_k u_k). \tag{5.44}$$

This gives

$$(P_k + B_k^T R_{k+1} B_k) u_k = -B_k^T R_{k+1} A_k x_k. \tag{5.45}$$

This gives the following nice expression for the optimal control

$$u_k^* = G_k x_k, \tag{5.46}$$

$$G_k = -(P_k + B_k^T R_{k+1} B_k)^{-1} B_k^T R_{k+1} A_k. \tag{5.47}$$

R_{k+1} may be found from the Riccati equation (5.32) or (5.33). However, we will in the next section derive a 3rd formulation of the discrete time Riccati equation which is to be preferred compared to Equations (5.32) and (5.33).

5.2.3 Alternative formulations of the discrete time Riccati equation

The discrete time Riccati equation in the LQ optimal control solution may be formulated in different ways. In Section (5.2) we have derived two different formulations. Se Equations (5.32) and (5.33). We will in this section propose two different formulations which does not involve the inversion of the weighting matrix P_k. These formulations are may be the most used formulations.

The starting point is as shown earlier, i.e., by putting Equation (5.18) (i.e. $p_k = Qx_k + A^T p_{k+1}$) into equation (5.26) (i.e. $p_k = R_k x_k$), we have

$$R_k x_k = Q_k x_k + A_k^T R_{k+1} x_{k+1}, \tag{5.48}$$

where we have used that at $p_{k+1} = R_{k+1} x_{k+1}$.

An expression for the closed loop system is obtained by putting the optimal control (5.46) and (5.47) into the discrete time state Equation $x_{k+1} = A_k x_k + B_k u_k$. This gives

$$x_{k+1} = (A_k - B_k(P_k + B_k^T R_{k+1} B_k)^{-1} B_k^T R_{k+1} A_k) x_k. \tag{5.49}$$

Putting (5.49) into (5.48) gives

$$R_k x_k = Q_k x_k + A_k^T R_{k+1}(A_k - B_k(P_k + B_k^T R_{k+1} B_k)^{-1} B_k^T R_{k+1} A_k) x_k. \tag{5.50}$$

This equation must hold for all states $x_k \neq 0$. Hence we have,

$$R_k = Q_k + A_k^T (R_{k+1} - R_{k+1} B_k (P_k + B_k^T R_{k+1} B_k)^{-1} B_k^T R_{k+1}) A_k. \tag{5.51}$$

This formulation of the discrete time Riccati equation is to be preferred. As we see, only the matrix $P_k + B_k^T R_{k+1} B_k$ have to be inverted. Note that the boundary condition is as before, i.e. $R_N = S_N$.

Finally, we will present a 4th formulation of the Riccati equation. Hence, we may show that

$$R_k = (A_k + B_k G_k)^T R_{k+1} (A_k + B_k G_k) + G_k^T P_k G_k + Q_k, \tag{5.52}$$

$$G_k = -(P_k + B_k^T R_{k+1} B_k)^{-1} B_k^T R_{k+1} A_k. \tag{5.53}$$

This formulation of the discrete time Riccati equation is known in the litterature as the Josephs stable version of the Riccati equation. As we see, this Riccati equation consists only of symmetric terms. This formulation is to be preferred in numerical calculations.

We also se that for a given control gain matrix, G_k, then Equation (5.52) is a discrete time Lyapunov equation. Equations (5.52) and (5.53) can with advantage be used in order to iterate to find the stationary solution to the LQ optimal control problem, i.e. the problem with infinite horizon $N \to \infty$.

Note that the boundary conditions to the different formulations of the Riccati equation is the same, i.e., $R_N = S_N$ where S_N is the weighting matrix for the final state, x_N.

Example 5.1 (MATLAB m-file: Solving the DARE)
The following m-file function solves the Discrete Algebraic Riccati equation (DARE) by iterating on the Josephs stable formulation of the Discrete Riccati Eqs. (5.52) and (5.53).

```
function [G,R]=dric_solv(A,B,S,Q,P,N);
% DRIC_SOLV
% [G,R]=dric_solv(A,B,S,Q,P,N);
% Iterate Riccati-equation backward
% in time from k=N+1 to k=1.

R=S;
for k=N:-1:1
 G = -inv(P+B'*R*B)*B'*R*A;
 R = (A+B*G)'*R*(A+B*G)+G'*P*G+Q;  % Josehps stabilized version of Riccati eq.
end
```

5.2.4 Numerical example

Example 5.2 (Singular transition matrix)
Given a system described by a linear discrete state space model with the following model matrices

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \; B = \begin{bmatrix} 0 \\ \sqrt{2} \end{bmatrix}, \; D = \begin{bmatrix} 1 & -1 \end{bmatrix}, \tag{5.54}$$

and with weighting matrices

$$P = 1, \; Q = D^T D = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \; S_N = Q. \tag{5.55}$$

We chose the following initial value for the state vector, i.e.,

$$x_i = \begin{bmatrix} x_{1,i} \\ x_{2,i} \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \tag{5.56}$$

and simulate the optimal closed loop system over the time horizon $i \leq k \leq N$ where $i = 0$ and $N = 5$. This gives after $N = 5$ iterations of the Riccati equation (5.53)

$$R_0 = \begin{bmatrix} 1 & -1 \\ -1 & 1.4993 \end{bmatrix}, R_1 = \begin{bmatrix} 1 & -1 \\ -1 & 1.497 \end{bmatrix}, R_2 = \begin{bmatrix} 1 & -1 \\ -1 & 1.488 \end{bmatrix}, \tag{5.57}$$

$$R_3 = \begin{bmatrix} 1 & -1 \\ -1 & 1.455 \end{bmatrix}, R_4 = \begin{bmatrix} 1 & -1 \\ -1 & 1.333 \end{bmatrix}, R_5 = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \tag{5.58}$$

and where $R_5 = S_5$ is defined from the specified final boundary value condition. It can be shown, se Pappas og Laub (1980), that the solution of the stationary discrete Riccati equation, i.e. the solution when $N \to \infty$, is given by

$$R = \begin{bmatrix} 1 & -1 \\ -1 & \frac{3}{2} \end{bmatrix}. \tag{5.59}$$

In general we have that $\lim_{N\to\infty} R_0 = R$. We se that even for a "short" horizon as $N = 5$ then R_0 is a relatively good approximation to the stationary solution, for this example.

Furthermore, the optimal time variant feedback matrices are given by

$$G_k = \begin{bmatrix} 0 & \frac{\sqrt{2}}{1+2r_{22,k+1}} \end{bmatrix} \;\; \forall\; k = 0, \ldots, 4 \tag{5.60}$$

where $r_{22,k+1}$ is the lower right element in R_{k+1}. This means that the optimal control is given by a feedback

$$u_k = \frac{\sqrt{2}}{1+2r_{22,k+1}} x_{2,k} \tag{5.61}$$

where $x_{2,k}$ is the 2nd state in the state vector (5.56). For this system it is optimal to only take feedback from one of the two states in the system. This is unusual because it in general is optimal with a feedback from all states in the system.

We remark that the system (A, B) is controllable and that (D, A) is observable. One special remark is that the system have two poles (eigenvalues) in origo. This means that the open loop system has infinite fast dynamics. The optimal system minimizes the objective J_i. The objective will in general obtain a small value if the state x_k goes fast to zero. It is therefore not optimal to make the system slower then necessary.

Simulations of the optimal control $u_k = G_k x_k$ and x_k is shown in Figure 5.1.

We end this example by mentioning that for systems with transport delay modeled as extra states, then the transition matrix will have eigenvalues in origo, and the optimal control will have a structure relatively equal to the above example.
△

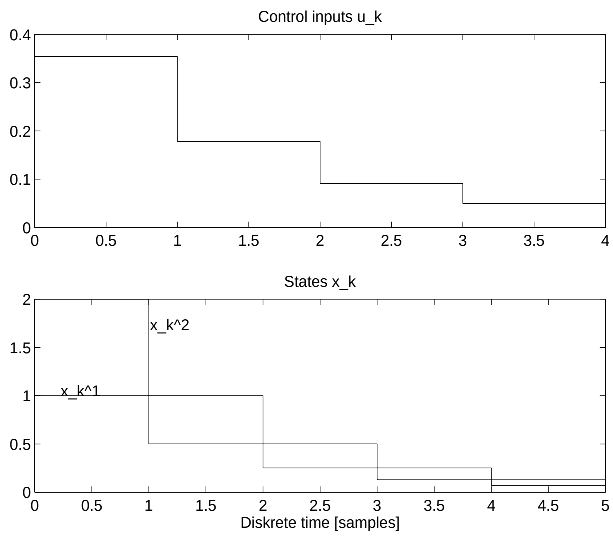

Figure 5.1: The Figure illustrates simulations of u_k and x_k for example 5.2. The discrete initial time is $i = 0$ and the final time instant is $N = 5$.

5.2.5 Summing up: The Discrete LQ (DLQ) optimal control problem

We will sum up some results in the following theorem

Theorem 5.2.1 (Discrete time Linear Quadratic optimal regulator) Given the discrete time system

$$x_{k+1} = A_k x_k + B_k u_k, \tag{5.62}$$

where $k \geq i$ and the initial value of the state vector, x_i, is given.

Consider given a LQ criterion valid over the time horizon $i \leq k \leq N$, i.e.,

$$J_i = \frac{1}{2} x_N^T S_N x_N + \frac{1}{2} \sum_{k=i}^{N-1} (x_k^T Q_k x_k + u_k^T P_k u_k), \tag{5.63}$$

where S_N, Q_k and P_k are symmetric weighting matrices.

The optimal control vector, u_k^*, which is minimizing the LQ criterion, J_i, is given by

$$u_k = G_k x_k, \tag{5.64}$$

$$G_k = -(P_k + B_k^T R_{k+1} B_k)^{-1} B_k^T R_{k+1} A_k, \tag{5.65}$$

where R_{k+1} is the positive solution to the Discrete time Riccati equation

$$R_k = Q_k + A_k^T (R_{k+1} - R_{k+1} B_k (P_k + B_k^T R_{k+1} B_k)^{-1} B_k^T R_{k+1}) A_k, \tag{5.66}$$

with final value boundary condition

$$R_N = S_N. \tag{5.67}$$

Furthermore, the minimum value of the criterion, J_i, is given by

$$J_i = \frac{1}{2} x_i^T R_i x_i. \tag{5.68}$$

and where R_i is found from the Riccati equation. $\triangle$

Merknad 5.1 *In some references it is common to define the state feedback matrix as $K_k = -G_k$, and $u_k = -K_k x_k$ instead of $u_k = G_k x_k$ as in these lecture notes. This is in particular the case as e.g. in Lewis and Syrmos (1995). The MATLAB Control System Toolbox also uses the notation $K = -G$, se e.g. the* **dlqr** *function.*

5.2.6 Standard Discrete time Linear Quadratic (DLQ) optimal control with infinite horizon

Theorem 5.2.2 (Discrete time Linear Quadratic (DLQ) optimal regulator)

Given a linear discrete time system

$$x_{k+1} = A x_k + B u_k, \tag{5.69}$$

$$y_k = D x_k, \tag{5.70}$$

where $k \geq i$ and the initial value of the state vector, x_i, is given.

Consider given a discrete time LQ criterion valid over an infinite horizon, i.e. time horizon $i \leq k \leq \infty$,

$$J_i = \frac{1}{2}\sum_{k=i}^{\infty}(y_k^T Q y_k + u_k^T P u_k) = \frac{1}{2}\sum_{k=i}^{\infty}(x_k^T \tilde{Q} x_k + u_k^T P u_k), \tag{5.71}$$

where $\tilde{Q} = D^T Q D$. Q and P are symmetric weighting matrices.

The optimal control vector, u_k^*, which is minimizing the LQ criterion, J_i, is given by

$$u_k^* = G x_k, \tag{5.72}$$
$$G = -(P + B^T R B)^{-1} B^T R A, \tag{5.73}$$

where $R > 0$ is the positive definite solution to the Discrete time Algebraic Riccati Equation(DARE)

$$R = D^T Q D + A^T R A - A^T R B (P + B^T R B)^{-1} B^T R A. \tag{5.74}$$

Furthermore, the minimum value of the criterion, J_i, is given by

$$J_i = \frac{1}{2} x_i^T R x_i. \tag{5.75}$$

and where $R > 0$ is found from the DARE. $\triangle$

Notice that the solution R_k to the discrete Riccati equation in Theorem 5.2.1 usually converges in a few number of samples to the solution R of the Discrete Algebraic Riccati Equation (DARE) in Theorem 5.2.2. This is illustrated as in e.g. Example 5.2.

5.2.7 Linear Quadratic (DLQ) optimal control: Cross weighting in the objective

Theorem 5.2.3 (DLQ optimal regulator: cross weighting)
Given a linear discrete time system

$$x_{k+1} = A x_k + B u_k, \tag{5.76}$$
$$y_k = D x_k + E u_k, \tag{5.77}$$

where $k \geq i$ and the initial value of the state vector, x_i, is given.

Consider given a discrete time LQ criterion valid over an infinite horizon, i.e. time horizon $i \leq k \leq \infty$,

$$J_i = \frac{1}{2}\sum_{k=i}^{\infty}(y_k^T Q y_k + u_k^T P u_k), \tag{5.78}$$

which gives the following objective with a cross weighting between x_k and u_k (here we have used the output equation Eq. (5.77) in the objective (5.78))

$$J_i = \frac{1}{2}\sum_{k=i}^{\infty}(x_k^T \tilde{Q} x_k + 2 x_k^T N u_k + u_k^T \tilde{P} u_k), \tag{5.79}$$

where $\tilde{Q} = D^T QD$, $N = D^T QE$ and $\tilde{P} = P + E^T QE$. Q and P are symmetric weighting matrices.

The optimal control vector, u_k^*, which is minimizing the LQ criterion, J_i, is given by

$$u_k^* = Gx_k, \tag{5.80}$$
$$G = -(P + B^T RB)^{-1}(B^T RA + N^T), \tag{5.81}$$

where $R > 0$ is the positive definite solution to the Discrete time Algebraic Riccati Equation(DARE)

$$R = D^T QD + A^T RA - (A^T RB + N)(P + B^T RB)^{-1}(B^T RA + N^T). \tag{5.82}$$

△

Furthermore, notice that the results in Theorem 5.2.3 are identical with the results in Theorem 5.2.2 when the cross weighting matrix $N = 0$ in the objective, and or when $E = 0$.

Theorem 5.2.3 may be proved from the Maximum principle. The Hamiltonian function of the objective in Eq. (5.79) and the state equation (5.76) is

$$\begin{aligned} H_k &= \frac{1}{2}(x_k^T Qx_k + 2x_k^T Nu_k + u_k^T Pu_k) + p_{k+1}^T(x_{k+1} - x_k) \\ &= \frac{1}{2}(x_k^T Qx_k + 2x_k^T Nu_k + u_k^T Pu_k) + p_{k+1}^T((A - I)x_k + Bu_k). \end{aligned} \tag{5.83}$$

The optimal control is found from

$$\frac{\partial H_k}{\partial u_k} = N^T x_k + Pu_k + B^T p_{k+1} = 0. \tag{5.84}$$

Using that $p_k = Rx_k$ gives

$$N^T x_k + Pu_k + B^T Rx_{k+1} = N^T x_k + Pu_k + B^T R(Ax_k + Bu_k) = 0. \tag{5.85}$$

This gives

$$(P + B^T RB)u_k = -(B^T RA + N^T)x_k, \tag{5.86}$$

and the optimal control

$$u_k^* = -(P + B^T RB)^{-1}(B^T RA + N^T)x_k, \tag{5.87}$$

The Riccati equation is derived from

$$p_{k+1} - p_k = -\frac{\partial H_k}{\partial x_k}. \tag{5.88}$$

This gives

$$p_{k+1} - p_k = -(Qx_k + Nu_k + (A^T - I)p_{k+1}) \tag{5.89}$$

and

$$p_k = Qx_k + A^T p_{k+1} + Nu_k. \tag{5.90}$$

This gives

$$p_k = Qx_k + A^T R(Ax_k + Bu_k) + Nu_k = Qx_k + A^T RAx_k + (A^T RB + N)u_k. \quad (5.91)$$

Using the optimal control and the relation $p_k = Rx_k$.

$$Rx_k = Qx_k + A^T RAx_k - (A^T RB + N)(P + B^T RB)^{-1}(B^T RA + N^T)x_k. \quad (5.92)$$

This equation must hold for all non trivial $x_k \neq 0$ and we have

$$R = Q + A^T RA - (A^T RB + N)(P + B^T RB)^{-1}(B^T RA + N^T), \quad (5.93)$$

which is Eq. (5.82) in Theorem 5.2.3.

5.2.8 Discrete LQ optimal control objective: Compact formulation

Notice that the LQ terms in the objective may be written more compact as discussed in the following. The function under the summation in the discrete LQ objective in Sec. 5.1 and Eq. (5.2) is

$$\begin{aligned}
L(x_k, u_k) &= \begin{bmatrix} x_k^T & u_k^T \end{bmatrix} \begin{bmatrix} Q & \frac{1}{2}M \\ \frac{1}{2}M^T & P \end{bmatrix} \begin{bmatrix} x_k \\ u_k \end{bmatrix} \\
&= \begin{bmatrix} x_k^T Q + \frac{1}{2}u_k^T M^T & \frac{1}{2}x_k^T M + u_k^T P \end{bmatrix} \begin{bmatrix} x_k \\ u_k \end{bmatrix} \\
&= x_k^T Q x_k + \frac{1}{2}u_k^T M^T x_k + \frac{1}{2}x_k^T M u_k + u_k^T P u_k \\
&= x_k^T Q x_k + x_k^T M u_k + u_k^T P u_k, \quad (5.94)
\end{aligned}$$

because $\frac{1}{2}u_k^T M^T x_k$ and $\frac{1}{2}x_k^T M u_k$ are scalars and then $\frac{1}{2}u_k^T M^T x_k + \frac{1}{2}x_k^T M u_k = x_k^T M u_k$.

5.3 Optimal tracking in discrete time systems

Given a system described by a linear discrete time state space model

$$\begin{aligned}
x_{k+1} &= A_k x_k + B_k u_k + C r_k, \quad (5.95) \\
y_k &= D x_k, \quad (5.96)
\end{aligned}$$

where $k \geq i$ is discrete time and the initial state x_i is given. $x_k \in \mathbb{R}^n$ is the state vector, $u_k \in \mathbb{R}^m$ is the control input vector and $y_k \in \mathbb{R}^m$ is the output vector.

We want the output, y_k, to be as close as possible to a known reference vector, r_k. In this case it make sense to use a control input, u_k, which minimize a control objective where the deviation $r_k - y_k$ is weighted in the objective. But control action costs so the control input, u_k, is also weighted in the objective.

We study the following control objective (ore performance index).

$$J_i = \frac{1}{2}(r_N - Dx_N)^T S_N (r_N - Dx_N) + \frac{1}{2}\sum_{k=i}^{N-1}[(r_k - Dx_k)^T Q_k (r_k - Dx_k) + u_k^T P_k u_k] \tag{5.97}$$

where $S_N \in \mathbb{R}^{m\times m}$, $Q_k \in \mathbb{R}^{m\times m}$ and $P_k \in \mathbb{R}^{r\times r}$, is symmetric weighting matrices.

Note that the reference vector, r_k, is influencing in the state equation (5.95).

usually, $C = 0$, but if we want integral action in the control system then an integrator for the deviation $r_k - y_k$ may be augmented in the model and a model of the form (5.95) is the result. The optimal control consist of a feedback from the complete state vector. Assume that a state equation of the form $x_{k+1} = A_k x_k + B_k u_k$ is augmented with an integrator $z_{k+1} = z_k + e_k$ where $e_k = r_k - y_k$ then the result is a state space model of the form as in Equation (5.95) with $C \neq 0$.

The state equation Equation (5.95) may be written as

$$x_{k+1} - x_k = (A_k - I)x_k + B_k u_k + C r_k. \tag{5.98}$$

The Hamiltonian function is then given by

$$H_k = \frac{1}{2}[(r_k - Dx_k)^T Q_k (r_k - Dx_k) + u_k^T P_k u_k] + p_{k+1}^T[(A_k - I)x_k + B_k u_k + C r_k]. \tag{5.99}$$

A 1st order confition for the existence of an optimal control vector, u_k^*, which minimizes the performance index J_i with the state space model as condition is that

$$\frac{\partial H_k}{\partial u_k} = P_k u_k + B_k^T p_{k+1} = 0. \tag{5.100}$$

We will firthermore assume the following relationship between the impulse vector, p_k, and the state vector, x_k, i.e.,

$$p_k = R_k x_k + h_k, \tag{5.101}$$

where $R_k \in \mathbb{R}^{n\times n}$ is an unknown matrix and where $h_k \in \mathbb{R}^n$ is an unknown n-dimensional vector.

Putting (5.101) into (5.100) gives

$$P_k u_k + B_k^T R_{k+1} x_{k+1} + B_k^T h_{k+1} = 0. \tag{5.102}$$

Substituting the state equation into this expression gives

$$P_k u_k + B_k^T R_{k+1}(A_k x_k + B_k u_k + C r_k) + B_k^T h_{k+1} = 0. \tag{5.103}$$

Solving with respect to u_k gives

$$u_k = -(P_k + B_k^T R_{k+1} B_k)^{-1}(B_k^T R_{k+1} A_k x_k + B_k^T R_{k+1} C r_k + B_k^T h_{k+1}). \tag{5.104}$$

From the maximum principle we have that the impulse vector is given by

$$p_{k+1} - p_k = -\frac{\partial H_k}{\partial x_k} = -D^T Q_k D x_k + D^T Q_k r_k - (A_k - I)^T p_{k+1}. \tag{5.105}$$

This may be simplified to

$$p_k = D^T Q_k D x_k - D^T Q_k r_k + A_k^T p_{k+1}. \tag{5.106}$$

Using the relationship $p_k = R_k x_k + h_k$ gives

$$\begin{aligned} R_k x_k + h_k &= D^T Q_k D x_k - D^T Q_k r_k + A_k^T p_{k+1} \\ &= D^T Q_k D x_k - D^T Q_k r_k + A_k^T R_{k+1} x_{k+1} + A_k^T h_{k+1}. \end{aligned} \tag{5.107}$$

We can now find an expression for x_{k+1} as a function of x_k by substituting the expression for the optimal control equation (5.104) into the state equation, equation (5.95). For simplicity we write the optimal control as follows

$$u_k^* = G_1 x_k + G_2 C r_k + G_3 h_{k+1}, \tag{5.108}$$
$$G_1 = -(P_k + B_k^T R_{k+1} B_k)^{-1} B_k^T R_{k+1} A_k, \tag{5.109}$$
$$G_2 = -(P_k + B_k^T R_{k+1} B_k)^{-1} B_k^T R_{k+1}, \tag{5.110}$$
$$G_3 = -(P_k + B_k^T R_{k+1} B_k)^{-1} B_k^T. \tag{5.111}$$

Hence, we have the following expression for the closed loop system

$$x_{k+1} = (A + BG_1) x_k + (BG_2 + I) C r_k + BG_3 h_{k+1}. \tag{5.112}$$

Putting Equation (5.112) into Equation (5.107) gives

$$\begin{aligned} R_k x_k + h_k = D^T Q_k D x_k - D^T Q_k r_k + A_k^T h_{k+1} \\ + A_k^T R_{k+1} [(A + BG_1) x_k + (BG_2 + I) C r_k + BG_3 h_{k+1}] \ . \end{aligned} \tag{5.113}$$

This may be written as follows

$$\begin{gathered} [-R_k + D^T Q_k D + A_k^T R_{k+1} (A + BG_1)] x_k + \\ [-h_k - D^T Q_k r_k + A_k^T h_{k+1} + A_k^T R_{k+1} (B_k G_2 + I) C r_k + A_k^T R_{k+1} B_k G_3 h_{k+1}]. \end{gathered} \tag{5.114}$$

This equation must hold for all $x_k \neq 0$. In order for this to hold the expressions in the brackets have to be zero. We have the equations for R_k and h_{k+1}, i.e.,

$$R_k = D^T Q_k D + A_k^T R_{k+1} (A + BG_1), \tag{5.115}$$

and

$$h_k = (A + BG_1)^T h_{k+1} - D^T Q_k r_k + A_k^T R_{k+1} (B_k G_2 + I) C r_k. \tag{5.116}$$

Equation (5.115) is the famous discrete time Riccati equation. Equation (5.116) is a difference equation for the feedforward signal h_k due to the external reference signal r_k. Equations (5.115) and(5.116) is solved backward in time from the final time instant, $k = N$. This means that we have to know some border conditions at the final time instant. This is discussed in the next section.

5.3.1 Border conditions

From the Maximum principle we have the border conditions

$$p_N = \frac{\partial}{\partial x_N}\left[\tfrac{1}{2}(r_N - Dx_N)^T S_N (r_N - Dx_N)\right], \tag{5.117}$$

which is equivalent with

$$p_N = \frac{\partial}{\partial x_N}\left[\tfrac{1}{2} r_N^T S_N r_N - r_N^T S_N D x_N + \tfrac{1}{2} x_N^T D^T S_N D x_N\right]. \tag{5.118}$$

Derivation gives

$$p_N = D^T S_N D x_N - D^T S_N r_N. \tag{5.119}$$

Expressing Equation (5.101) at time $k = N$ gives

$$p_N = R_N x_N + h_N. \tag{5.120}$$

Comparing Equations (5.119) and (5.120) gives us the final time (value) conditions

$$R_N = D^T S_N D, \tag{5.121}$$

$$h_N = -D^T S_N r_N. \tag{5.122}$$

5.3.2 Summary

The results in this section is summed up in the following Theorem 5.3.1.

Theorem 5.3.1 (Optimal tracking in discrete time systems)
Given a discrete time state space model

$$x_{k+1} = A_k x_k + B_k u_k + C r_k, \tag{5.123}$$

$$y_k = D x_k, \tag{5.124}$$

and a Linear Quadratic (LQ) control objective (performance index) defined over the finite time horizon $i \leq k \leq N$

$$\begin{aligned} J_i = {} & \tfrac{1}{2}(r_N - Dx_N)^T S_N (r_N - Dx_N) \\ & + \tfrac{1}{2}\textstyle\sum_{k=i}^{N-1}[(r_k - Dx_k)^T Q_k (r_k - Dx_k) + u_k^T P_k u_k], \end{aligned} \tag{5.125}$$

where $S_N \in \mathbb{R}^{m\times m}$, $Q_k \in \mathbb{R}^{m\times m}$ and $P_k \in \mathbb{R}^{r\times r}$ are symmetric positive semi-definite weighting matrices.

The optimal control which minimizes the objective J_i is given by

$$u_k^* = G_1 x_k + G_2 C r_k + G_3 h_{k+1}, \tag{5.126}$$

$$G_1 = -(P_k + B_k^T R_{k+1} B_k)^{-1} B_k^T R_{k+1} A_k, \tag{5.127}$$

$$G_2 = -(P_k + B_k^T R_{k+1} B_k)^{-1} B_k^T R_{k+1}, \tag{5.128}$$

$$G_3 = -(P_k + B_k^T R_{k+1} B_k)^{-1} B_k^T. \tag{5.129}$$

R_{k+1} is the solution to the discrete Riccati-equation,

$$R_k = D^T Q_k D + A_k^T R_{k+1}(A + BG_1), \tag{5.130}$$

and the feed forward signal vector h_{k+1} is given from the difference equation

$$h_k = (A + BG_1)^T h_{k+1} - D^T Q_k r_k + A_k^T R_{k+1}(B_k G_2 + I)C r_k. \tag{5.131}$$

The border conditions is at the final time instant $k = N$ given by

$$R_N = D^T S_N D, \tag{5.132}$$
$$h_N = -D^T S_N r_N. \tag{5.133}$$

$\triangle$

Theorem 5.3.2 (Optimal tracking: Minimum of the objective J_i)
Given the state space model, Equations (5.123) and (5.124) mith $C = 0$. Given the solution to the LQ optimal control problem as presented in Theorem 5.3.1.

The minimum of the control objective (performance index), equation (5.125) over the discrete time horizon $i \leq k < N$ where i is the initial time, is given by

$$J_k^* = \frac{1}{2} x_k^T R_k x_k + x_k^T h_k + w_k, \tag{5.134}$$

where h_k is given by Equation (5.131) and where the signal w_k satisfies the difference-equation

$$w_k = w_{k+1} + \frac{1}{2} r_k^T Q_k r_k - \frac{1}{2} h_{k+1}^T B_k (B_k^T R_{k+1} B_k + P_k)^{-1} B_k^T h_{k+1}, \tag{5.135}$$

with border conditions at the final time instant given by

$$w_N = \frac{1}{2} r_N^T S_N r_N. \tag{5.136}$$

$\triangle$

5.4 Weighting control deviations in the LQ objective

5.4.1 Standard LQ control and weighting control deviations

Assume given a system described by a linear discrete time state space model

$$x_{k+1} = A_k x_k + B_k u_k, \tag{5.137}$$
$$y_k = D x_k. \tag{5.138}$$

Consider the problem of minimizing the LQ objective

$$J_i = \frac{1}{2} y_N^T S_N y_N + \frac{1}{2} \sum_{k=i}^{N-1} (y_k^T Q_k y_k + \Delta u_k^T \mathcal{R}_k \Delta u_k) \tag{5.139}$$

with respect to the control deviations $\Delta u_k \; \forall \; k = 1, \ldots, N-1$.

Notice that we now have the choice of formulating the problem in terms of deviation variables $\Delta u_k = u_k - u_{k-1}$ or in terms of actual control variables u_k. We chose to formulate the problem in terms of control input deviations Δu_k. The two alternatives gives the same results anyway.

the problem may be reformulated as a standard LQ optimal control problem. We start by augmenting the process model eq. (5.137) with $u_k = u_{k-1} + \Delta u_k$. This gives the augmented state space model

$$\overbrace{\begin{bmatrix} x_{k+1} \\ u_k \end{bmatrix}}^{\tilde{x}_{k+1}} = \overbrace{\begin{bmatrix} A_k & B_k \\ 0_{r\times n} & I_{r\times r} \end{bmatrix}}^{\tilde{A}_k} \overbrace{\begin{bmatrix} x_k \\ u_{k-1} \end{bmatrix}}^{\tilde{x}_k} + \overbrace{\begin{bmatrix} B_k \\ I_{r\times r} \end{bmatrix}}^{\tilde{B}_k} \Delta u_k \tag{5.140}$$

$$y_k = \overbrace{\begin{bmatrix} D & 0_{m\times r} \end{bmatrix}}^{\tilde{D}} \overbrace{\begin{bmatrix} x_k \\ u_{k-1} \end{bmatrix}}^{\tilde{x}_k} \tag{5.141}$$

where $0_{n\times r}$ and $0_{m\times r}$ is an $n \times r$ matrix and $m \times r$ matrix with zeroes, respectively. $I_{r\times r}$ is an $r \times r$ identity matrix.

The LQ criterion may be written as

$$J_i = \frac{1}{2} \begin{bmatrix} x_N \\ u_{N-1} \end{bmatrix}^T \overbrace{\begin{bmatrix} D^T S_N D & 0 \\ 0 & 0 \end{bmatrix}}^{\tilde{S}_N} \begin{bmatrix} x_N \\ u_{N-1} \end{bmatrix}$$
$$+ \sum_{k=i}^{N-1} (\begin{bmatrix} x_k \\ u_{k-1} \end{bmatrix}^T \overbrace{\begin{bmatrix} D^T Q_k D & 0 \\ 0 & 0 \end{bmatrix}}^{\tilde{Q}_k} \begin{bmatrix} x_k \\ u_{k-1} \end{bmatrix} + \Delta u_k^T \mathcal{R}_k \Delta u_k) \tag{5.142}$$

We find the solution to this LQ optimal control problem by using the results in Theorem 5.2.1 but with model matrices $\tilde{A}_k$ and $\tilde{B}_k$ and with weighting matrices $\tilde{S}_N$, $\tilde{Q}_k$ and $P = \mathcal{R}$. This results in the state feedback matrix $\tilde{G}_k$.

The optimal control deviation is then given by

$$\Delta u_k = \overbrace{\begin{bmatrix} G_1 & G_2 \end{bmatrix}_k}^{\tilde{G}_k} \begin{bmatrix} x_k \\ u_{k-1} \end{bmatrix} = G_1 x_k + G_2 u_{k-1}. \tag{5.143}$$

The actual optimal control to the process is given by

$$u_k = \overbrace{G_1 x_k + G_2 u_{k-1}}^{\Delta u_k} + u_{k-1} = G_1 x_k + (G_2 + I_{r\times r}) u_{k-1}. \tag{5.144}$$

This optimal controller will among others, if we increase the weight $\mathcal{R}$ on the control deviations Δu_k will give a smother control action u_k. The problem presented in this section can with advantage be extended to weighting the control deviation $y_k - r_k$ where r_k is a specified reference signal.

In order to get grater insight into the solution of this problem, we may with advantage use the maximum principle directly.

5.4.2 Optimal tracking and weighting control deviations

An LQ objective which make sense in the case where both the output y_k and the control u_k have steady state values different from zero is as follows

$$J_i = \frac{1}{2}(r_N - y_N)^T S_N (r_N - y_N) + \frac{1}{2}\sum_{k=i}^{N-1}((r_k - y_k)^T Q_k (r_k - y_k) + \Delta u_k^T \mathcal{R}_k \Delta u_k). \quad (5.145)$$

Using the augmented model (5.140) and (5.141) we find that the above objective may be written as

$$\begin{aligned} J_i = \tfrac{1}{2}(r_N - \tilde{D}\tilde{x}_N)^T S_N (r_N - \tilde{D}\tilde{x}_N) \\ +\tfrac{1}{2}\textstyle\sum_{k=i}^{N-1}((r_k - \tilde{D}\tilde{x}_k)^T Q_k (r_k - \tilde{D}\tilde{x}_k) + \Delta u_k^T \mathcal{R}_k \Delta u_k). \end{aligned} \quad (5.146)$$

The solution to this LQ optimal control tracking problem is given as presented in Theorem 5.3.1.

We present the result in the following theorem. However, notice that we have added the matrix $\tilde{C}$ in the problem solution for the sake of completeness.

Theorem 5.4.1 (Weighting control deviations and optimal tracking)
Given the discrete time state space model

$$\tilde{x}_{k+1} = \tilde{A}_k \tilde{x}_k + \tilde{B}_k \Delta u_k + \tilde{C} r_k, \quad (5.147)$$

$$y_k = \tilde{D}\tilde{x}_k, \quad (5.148)$$

However, notice that we have added the term $\tilde{C} r_k$ in the model, for the sake of completeness of the solution, and notice that the model (5.137) and (5.137) does not have a term $C r_k$.

Given an LQ objective defined over the time interval $i \leq k \leq N$

$$\begin{aligned} J_i = \tfrac{1}{2}(r_N - \tilde{D}\tilde{x}_N)^T S_N (r_N - \tilde{D}\tilde{x}_N) \\ +\tfrac{1}{2}\textstyle\sum_{k=i}^{N-1}((r_k - \tilde{D}\tilde{x}_k)^T Q_k (r_k - \tilde{D}\tilde{x}_k) + \Delta u_k^T \mathcal{R}_k \Delta u_k). \end{aligned} \quad (5.149)$$

where $S_N \in \mathbb{R}^{m \times m}$, $Q_k \in \mathbb{R}^{m \times m}$ and $P_k \in \mathbb{R}^{r \times r}$, are symmetric weighting matrices.

The optimal control which is minimizing the objective J_i is given by

$$\Delta u_k = G_1 \tilde{x}_k + G_2 \tilde{C} r_k + G_3 h_{k+1}, \quad (5.150)$$

$$G_1 = -(\mathcal{R}_k + \tilde{B}_k^T R_{k+1} \tilde{B}_k)^{-1} \tilde{B}_k^T R_{k+1} \tilde{A}_k, \quad (5.151)$$

$$G_2 = -(\mathcal{R}_k + \tilde{B}_k^T R_{k+1} \tilde{B}_k)^{-1} \tilde{B}_k^T R_{k+1}, \quad (5.152)$$

$$G_3 = -(\mathcal{R}_k + \tilde{B}_k^T R_{k+1} \tilde{B}_k)^{-1} \tilde{B}_k^T. \quad (5.153)$$

R_{k+1} is the solution to the discrete time Riccati equation

$$R_k = \tilde{D}^T Q_k \tilde{D} + \tilde{A}_k^T R_{k+1}(\tilde{A} + \tilde{B} G_1), \quad (5.154)$$

and the feed-forward signal h_{k+1} is given by the difference equation

$$h_k = (\tilde{A} + \tilde{B} G_1)^T h_{k+1} - \tilde{D}^T Q_k r_k + \tilde{A}_k^T R_{k+1}(\tilde{B}_k G_2 + I)\tilde{C} r_k. \quad (5.155)$$

The border conditions (final value conditions) at the final time $k = N$ is given by

$$R_N = \tilde{D}^T S_N \tilde{D}, \tag{5.156}$$
$$h_N = -\tilde{D}^T S_N r_N. \tag{5.157}$$

△

An alternative suboptimal strategy to the one presented in Theorem 5.4.1 is to use the solution to the discrete algebraic Riccati equation (DARE), i.e. with $R = R_k = R_{k+1}$. The difference equation for computing the feed-forward signal h_k is as before and given by 5.155, but with $R_{k+1} = R$, G_1 and G_2 are constant feedback matrices. This strategy is in many cases to be preferred because it simplify the solution considerably and the difference are in many cases minor.

One should also notice the alternative final value condition h_N which with advantage could be used in this case, i.e. the steady state solution to (5.155), i.e.,

$$G = -(\mathcal{R}_N + \tilde{B}_N^T R \tilde{B}_N)^{-1} \tilde{B}_N^T R \tilde{A}_N, \tag{5.158}$$
$$h_N = (I - (\tilde{A}_N + \tilde{B}_N G)^T)^{-1}(-\tilde{D}^T Q + \tilde{A}_N^T R(\tilde{B}_N G_2 + I)C) r_N. \tag{5.159}$$

This final value condition ensures integral action and zero steady state error at the final time.

It is of importance to illustrate the implementation of this strategy. In connection to this we refer to the MATLAB script file **main_dlq_rdu.m**.

A modified version where we are using a mowing horizon control strategy as in Model Predictive Control (MPC) is given in the file **main_dlq_rdu2.m**.

Example 5.3 (Weighting control deviations)
Given the system

$$x_{k+1} = Ax_k + Bu_k, \tag{5.160}$$
$$y_k = Dx_k, \tag{5.161}$$

where

$$A = \begin{bmatrix} 1.5 & 1.0 & 0.10 \\ -0.7 & 0 & 0.10 \\ 0 & 0 & 0.85 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 3 & 0 & -0.6 \\ 0 & 1 & 1 \end{bmatrix}. \tag{5.162}$$

We specify the following weighting matrices

$$Q = \begin{bmatrix} 0.03 & 0 \\ 0 & 0.03 \end{bmatrix}, \quad \mathcal{R} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{5.163}$$

Solving the DARE gives

$$R = \begin{bmatrix} 1.5873 & 1.0940 & -0.0290 & -0.2046 & 0.3320 \\ 1.0940 & 0.9971 & 0.0963 & -0.1657 & 0.4078 \\ -0.0290 & 0.0963 & 0.1033 & 0.0623 & 0.1344 \\ -0.2046 & -0.1657 & 0.0623 & 0.3757 & 0.0140 \\ 0.3320 & 0.4078 & 0.1344 & 0.0140 & 0.7163 \end{bmatrix}, \tag{5.164}$$

$$G = \begin{bmatrix} 0.2046 & 0.1657 & -0.0623 & -0.3757 & -0.0140 \\ -0.3320 & -0.4078 & -0.1344 & -0.0140 & -0.7163 \end{bmatrix}. \tag{5.165}$$

This example is implemented in the MATLAB m-file **main_dlq_rdu.m** *and* **main_dlq_rdu2.m**. *Executing the files gives the results as illustrated in Figures 5.2 and 5.3.*

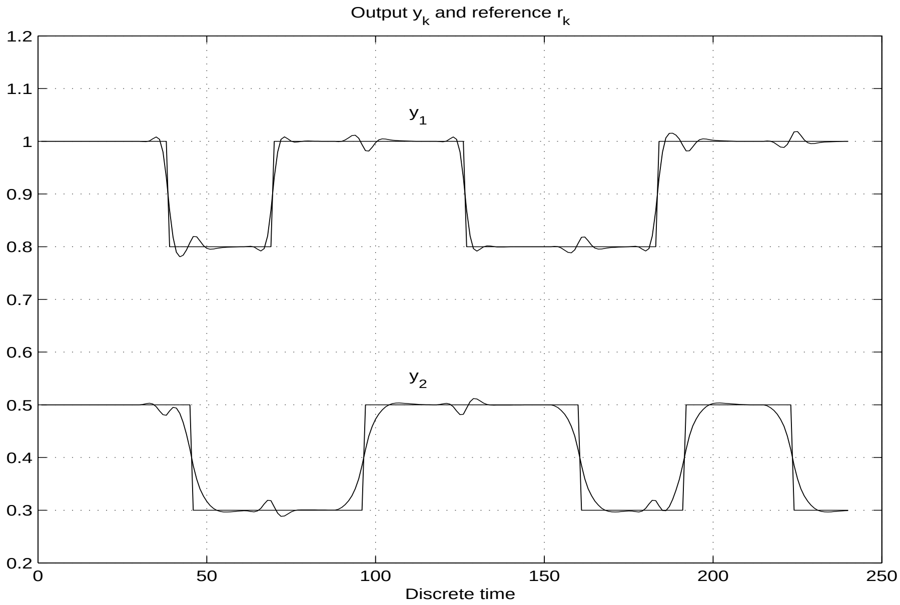

Figure 5.2: Simulation of the system in Example 10.1. this Figure is generated by executing the MATLAB script-file **main_dlq_rdu2.m**.

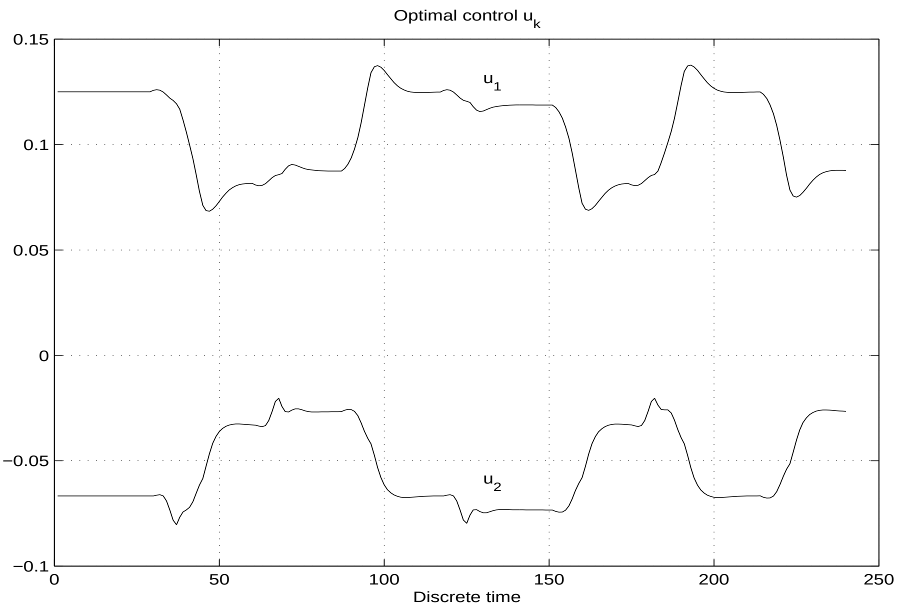

Figure 5.3: Simulation of the system in Example 10.1. this Figure is generated by executing the MATLAB script-file **main_dlq_rdu2.m**.

5.5 LQ control objective used in MPC

We will in this section study the solution to the LQ optimal control problem where $r_k - y_k$, Δu_k and u_k are weighted in the control objective. This objective is also used by the EMPC algorithm.

Consider a discrete time linear process model

$$x_{k+1} = A_k x_k + B_k u_k, \tag{5.166}$$
$$y_k = D x_k. \tag{5.167}$$

and a performance index

$$\begin{aligned} J_i &= \frac{1}{2}(r_N - y_N)^T S_N (r_N - y_N) \\ &+ \frac{1}{2}\sum_{k=i}^{N-1}((r_k - y_k)^T Q_k (r_k - y_k) + \Delta u_k^T \mathcal{R}_k \Delta u_k + u_k^T P u_k), \end{aligned} \tag{5.168}$$

where S_N, Q_k, $\mathcal{R}_k$ and P_k are weighting matrices. We will in the following use the maximum principle in order to derive the optimal control.

5.5.1 Computing u_k^*

The Hamilton function is

$$\begin{aligned} H_k &= \frac{1}{2}((r_k - y_k)^T Q_k (r_k - y_k) + (u_k - u_{k-1})^T \mathcal{R}_k (u_k - u_{k-1}) + u_k^T P_k u_k) \\ &+ p_{k+1}^T((A_k - I)x_k + B_k u_k). \end{aligned} \tag{5.169}$$

The co-state

An equation for the co-state is

$$p_{k+1} - p_k = -\frac{\partial H_k}{\partial x_k} = -(-D^T Q_k (r_k - D x_k) + (A_k^T - I)p_{k+1}) \tag{5.170}$$

which gives

$$p_k = D^T Q_k D x_k + A_k^T p_{k+1} - D^T Q_k r_k \tag{5.171}$$

The optimal control

$$\frac{\partial H_k}{\partial u_k} = \mathcal{R}_k(u_k - u_{k-1}) + P_k u_k + B^T p_{k+1} = 0 \tag{5.172}$$

which gives

$$(\mathcal{R}_k + P_k)u_k = \mathcal{R}_k u_{k-1} - B_k^T p_{k+1}, \tag{5.173}$$

which can be solved for u_k if the matrix $\mathcal{R}_k + P_k$ is non-singular. However, we will in the following find an expression for u_k in terms of variables which is defined at time k only (not in terms of p_{k+1}).

In order to continue we will assume that there are a relationship

$$p_k = R_k x_k + h_k. \tag{5.174}$$

Substituting (5.174) into (5.173) gives

$$(\mathcal{R}_k + P_k)u_k = \mathcal{R}_k u_{k-1} - B_k^T(R_{k+1}x_{k+1} + h_{k+1}), \tag{5.175}$$

Substituting for the state x_{k+1} given by (5.166) gives

$$(\mathcal{R}_k + P_k)u_k = \mathcal{R}_k u_{k-1} - B_k^T R_{k+1}(A_k x_k + B_k u_k) - B_k^T h_{k+1}. \tag{5.176}$$

Solving for u_k gives

$$u_k = G_1 x_k + G_3 h_{k+1} + G_4 u_{k-1}, \tag{5.177}$$

where

$$G_1 = -(\mathcal{R}_k + P_k + B_k^T R_{k+1} B_k)^{-1} B_k^T R_{k+1} A_k \tag{5.178}$$
$$G_3 = -(\mathcal{R}_k + P_k + B_k^T R_{k+1} B_k)^{-1} B_k^T \tag{5.179}$$
$$G_4 = -(\mathcal{R}_k + P_k + B_k^T R_{k+1} B_k)^{-1} \mathcal{R}_k \tag{5.180}$$

The closed loop system

Substituting the optimal control into the process model gives

$$x_{k+1} = (A + BG_1)x_k + BG_3 h_{k+1} + BG_4 u_{k-1} \tag{5.181}$$

The Riccati equation and the feed-forward signal

Substituting (5.171) into (5.174) gives

$$D^T Q_k D x_k + A_k^T p_{k+1} - D^T Q_k r_k = R_k x_k + h_k \tag{5.182}$$

Using that $p_{k+1} = R_{k+1}x_{k+1} + h_{k+1}$ gives

$$D^T Q_k D x_k + A_k^T R_{k+1} x_{k+1} + A_k^T h_{k+1} - D^T Q_k r_k = R_k x_k + h_k \tag{5.183}$$

Substituting (5.181) for (x_{k+1}) gives

$$\begin{aligned} D^T Q_k D x_k + A_k^T R_{k+1}((A_k + B_k G_1)x_k + B_k G_3 h_{k+1} + B_k G_4 u_{k-1}) \\ + A_k^T h_{k+1} - D^T Q_k r_k = R_k x_k + h_k, \end{aligned} \tag{5.184}$$

which can be rewritten as

$$\begin{aligned} [-R_k + A_k^T R_{k+1}(A_k + B_k G_1) + D^T Q_k D]x_k \\ -h_k + (A_k^T + A_k^T R_{k+1}^T B_k G_3)h_{k+1} + A_k^T R_{k+1} B_k G_4 u_{k-1} - D^T Q_k r_k = 0 \end{aligned} \tag{5.185}$$

Equation (5.185) must hold for all x_k so that

$$R_k = A_k^T R_{k+1}(A_k + B_k G_1) + D^T Q_k D, \tag{5.186}$$

$$h_k = (A_k^T + A_k^T R_{k+1}^T B_k G_3) h_{k+1} + A_k^T R_{k+1} B_k G_4 u_{k-1} - D^T Q_k r_k. \tag{5.187}$$

Equation (5.186) is the well known discret time Riccati equation. Note that the difference equation for the feed-forward signal can be expressed as

$$h_k = (A_k + B_k G_1)^T h_{k+1} + A_k^T R_{k+1} B_k G_4 u_{k-1} - D^T Q_k r_k. \tag{5.188}$$

Final value conditions

We have similar conditions as in the standard tracking problem.

$$R_N = D^T S_N D \tag{5.189}$$

$$h_N = -D^T S_N r_N \tag{5.190}$$

5.5.2 Discussion

Equations (5.188) and (5.186) has to be iterated backwards from $k = N - 1$ to time $k = i$. One problem is here that u_{k-1} in (5.188) is not known for $k = i + 1, \ldots, k = N - 1$. Hence, a question is how to solve the problem.

5.6 Solution to the discrete algebraic Riccati equation (DARE)

Consider a discrete infinite time LQ optimal control problem, i.e. find the optimal control u_k for a system $x_{k+1} = Ax_k + Bu_k$ with performance index $J_i = \sum_{k=i}^{\infty}(x_k^T Q x_k + u_k^T P u_k)$ and where the pair (A, B) is stabilizable and where the pair $(\sqrt{Q}, A)$ is detectable.

From the maximum principle, i.e., $\frac{\partial H_k}{\partial u_k} = 0$, $x_{k+1} - x_k = \frac{\partial H_k}{\partial p_k}$ and $p_{k+1} - p_k = -\frac{\partial H_k}{\partial x_k}$, we have the two point boundary value problem

$$x_{k+1} = Ax_k - BP^{-1}B^T p_{k+1}, \tag{5.191}$$

$$p_k = Qx_k + A^T p_{k+1}, \tag{5.192}$$

with initial state x_i given and final co-state $p_\infty = 0$. Equations (5.191) and (5.192) can be written in matrix form as follows

$$\overbrace{\begin{bmatrix} I & BP^{-1}B^T \\ 0 & A^T \end{bmatrix}}^{F_1} \begin{bmatrix} x_{k+1} \\ p_{k+1} \end{bmatrix} = \overbrace{\begin{bmatrix} A & 0 \\ -Q & I \end{bmatrix}}^{F_2} \begin{bmatrix} x_k \\ p_k \end{bmatrix}. \tag{5.193}$$

Consider now the generalized eigenvalue problem

$$|F_1\lambda - F_2| = 0, \tag{5.194}$$

and the corresponding generalized eigenvalue and eigenvector problem

$$F_1 M \Lambda = F_2 M, \tag{5.195}$$

where M is the matrix of generalized eigenvectors and where Λ is the matrix of generalized eigenvalues. Equation (5.195) can be partitioned as follows

$$F_1 \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \begin{bmatrix} \Lambda_{11} & 0 \\ 0 & \Lambda_{22} \end{bmatrix} = F_2 \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}, \tag{5.196}$$

where Λ_{11} is a diagonal matrix with the n stable generalized eigenvalues and Λ_{22} contains the n unstable generalized eigenvalues.

From this we have that

$$R = M_{21}M_{11}^{-1} = A^T V_{21}\Lambda_{11}V_{11}^{-1} + Q \tag{5.197}$$

is a solution to the discrete ARE

$$R = A^T R(I + BP^{-1}B^T R)^{-1}A + Q. \tag{5.198}$$

This can be proved by substituting (5.197) and the equations obtained from (5.196) into the DARE (5.198). Similarly we can prove that the closed loop system is stable, i.e. that the closed loop system contains the eigenvalues in Λ_{11}.

Proof 5.1 (Solution to the DARE)
From (5.196) we have that

$$(M_{11} + BP^{-1}B^T M_{21})\Lambda_{11} = AM_{11}, \tag{5.199}$$
$$A^T M_{21}\Lambda_{11} = -QM_{11} + M_{21}. \tag{5.200}$$

Equation (5.199) gives

$$A = (I + BP^{-1}B^T M_{21}M_{11}^{-1})M_{11}\Lambda_{11}M_{11}^{-1}. \tag{5.201}$$

Using $R = M_{21}M_{11}^{-1}$ and substituting into the DARE (5.198) gives

$$\begin{aligned} R &= A^T R(I + BP^{-1}B^T R)^{-1}(I + BP^{-1}B^T M_{21}M_{11}^{-1})M_{11}\Lambda_{11}M_{11}^{-1} + Q \\ &= A^T RM_{11}\Lambda_{11}M_{11}^{-1} + Q = A^T M_{21}\Lambda_{11}M_{11}^{-1} + Q \end{aligned} \tag{5.202}$$

Substituting (5.200) into (5.202) gives

$$R = (-QM_{11} + M_{21})M_{11}^{-1} + Q = M_{21}M_{11}^{-1}. \tag{5.203}$$

This proves that $R = M_{21}M_{11}^{-1}$ is a solution to the DARE. **QED**.

Proof 5.2 (Stability of the closed loop system)
An expression for the closed loop system is given by (see Equation 4.25)

$$A_{cl} = (I + BP^{-1}B^T R)^{-1}A. \tag{5.204}$$

Substituting for A given by (5.201) into (5.204) gives

$$A_{cl} = M_{11}\Lambda_{11}M_{11}^{-1}, \tag{5.205}$$

which proves that the eigenvalues of the closed loop system is given by Λ_{11}. **QED**.

The generalized eigenvalue/eigenvector problem can be solved in MATLAB by $[M, \Lambda] = \mathrm{eig}(F_2, F_1)$. Note also that the Control Systems Toolbox function $[-G, R] = \mathrm{dlqr}(A, B, Q, P)$ does not work when A is singular. However, the above method works for singular transition matrices.

Example 5.4
Consider a system

$$x_{k+1} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} x_k + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_k \tag{5.206}$$
$$y_k = \begin{bmatrix} 1 & 0 \end{bmatrix} x_k \tag{5.207}$$

and the objective function

$$J_0 = \sum_{k=0}^{\infty}(y_k^T y_k + u_k^T u_k). \tag{5.208}$$

The problem is to find a solution to the discrete ARE and the optimal feedback gain.

First, note that we have weighting matrices $Q = D^T D = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ *and* $P = 1$.

The matrices in the generalized eigenvalue and eigenvector problem are

$$F_1 = \begin{bmatrix} I & BP^{-1}B^T \\ 0 & A^T \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \tag{5.209}$$

and

$$F_2 = \begin{bmatrix} A & 0 \\ -Q & I \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{5.210}$$

The MATLAB command $[\tilde{M}, \tilde{\Lambda}] = eig(F_2, F_1)$. *gives*

$$\tilde{M} = \begin{bmatrix} M_{12} & M_{11} \\ M_{22} & M_{21} \end{bmatrix} = \begin{bmatrix} 0 & 0.3887 & -0.4777 & 0.5774 \\ -0.7071 & 0.6290 & 0.2952 & -0.5774 \\ 0 & -0.2402 & -0.7730 & 0.5774 \\ 0.7071 & -0.6290 & -0.2952 & -0.0000 \end{bmatrix} \tag{5.211}$$

and

$$\tilde{\Lambda} = \begin{bmatrix} \Lambda_{22} & 0 \\ 0 & \Lambda_{11} \end{bmatrix} = \begin{bmatrix} -Inf + NaNi & 0 & 0 & 0 \\ 0 & 2.6180 & 0 & 0 \\ 0 & 0 & 0.3820 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{5.212}$$

Note that the three finite generalized eigenvalues $\lambda_2 = 2.618$, $\lambda_3 = 0.382$ *and* $\lambda_4 = 0$ *are the roots of the characteristic equation* $\det(F_1\lambda - F_2) = (-\lambda^2 + 3\lambda - 1)\lambda = 0$. *This can be partitioned according to (5.196), i.e. with the stable eigenvalues first. Hence, we have*

$$M_{11} = \begin{bmatrix} -0.4777 & 0.5774 \\ 0.2952 & -0.5774 \end{bmatrix} \tag{5.213}$$

and

$$M_{21} = \begin{bmatrix} -0.7730 & 0.5774 \\ -0.2952 & -0.0000 \end{bmatrix}. \tag{5.214}$$

This gives

$$R = M_{21} M_{11}^{-1} = \begin{bmatrix} 2.618 & 1.618 \\ 1.618 & 1.618 \end{bmatrix} \tag{5.215}$$

and the optimal feedback $u_k = G x_k$ *with optimal gain matrix*

$$G = -(P + B^T R B)^{-1} B^T R A = -\begin{bmatrix} 0.618 & 0.618 \end{bmatrix}. \tag{5.216}$$

Chapter 6

Discrete LQ optimal control: Alternative direct solution

6.1 The objective function

Lemma 6.1 (Discrete Linear Quadratic Regulator)
Consider the standard LQ performance index or objective function

$$J_i = \frac{1}{2}x_N^T S_N x_N + \frac{1}{2}\sum_{k=i}^{N-1}(x_k^T Q_k x_k + u_k^T P_k u_k), \tag{6.1}$$

where S_N, Q_k and P_k are symmetric weighting matrices. i is the discrete initial time instant and N the discrete final time instant.

The LQR optimal controller is given by

$$u_k^* = G_k x_k \tag{6.2}$$
$$G_k = -(P_k + B^T R_{k+1} B)^{-1} B^T R_{k+1} A \tag{6.3}$$

where R_k is the non-negative solution, for al time instants $i \leq k \leq N$, of the Riccati difference equation

$$R_k = A^T(R_{k+1} - R_{k+1}B(P + B^T R_{k+1} B)^{-1} B^T R_{k+1})A + Q_k, \tag{6.4}$$
$$R_N = S_N. \tag{6.5}$$

The minimum of the objective eq. (6.1) is given by

$$J_i^* = \frac{1}{2}x_i^T R_i x_i. \tag{6.6}$$

△

It is clear that the objective eq. (6.1) may be written as

$$J_i = \frac{1}{2}\sum_{k=i}^{N-1}(x_{k+1}^T Q_{k+1} x_{k+1} + u_k^T P_k u_k) + \frac{1}{2}x_i^T Q_i x_i \tag{6.7}$$

when $Q_N = S_N$. The reason for separating the term $\frac{1}{2}x_i^T Q_i x_i$ is that it can not be influenced upon by the unknown control actions $u_k \; \forall \; k = i, i+1, \ldots, N-1$.

Putting the initial time i equal to the actual present time instant k, and noticing that the objective functions (6.7) is defined at $L = N - i + 1$ discrete time instants, including the present time instant $k = i$, then we may write the objective eq. (6.7) as

$$J_k = \frac{1}{2}\sum_{i=1}^{L-1}(x_{k+i}^T Q_{k+i} x_{k+i} + u_{k+i-1}^T P_{k+i-1} u_{k+i-1}) + \frac{1}{2}x_k^T Q_k x_k \tag{6.8}$$

This objective function is usually used in connection with Model Predictive Control (MPC) and the prediction horizon is here defined as $L-1$. The objective functions eqs. (6.7) - (6.8) are equal when $L = N - i + 1$ and $L - 1 = N - i$.

Since the last term in (6.8) is not influenced by the unknown control actions we may instead minimize the performance index

$$J_k^{MPC} = \frac{1}{2}\sum_{i=1}^{T}(x_{k+i}^T Q_{k+i} x_{k+i} + u_{k+i-1}^T P_{k+i-1} u_{k+i-1}) \tag{6.9}$$

where here $T = L - 1 = N - i$ is the prediction horizon.

6.2 Compact description

The objective function eq. (6.8) may be written compact as follows

$$J_k = \frac{1}{2}(x_{k+1|L}^T Q_{k|L} x_{k+1|L} + u_{k|L}^T P_{k|L} u_{k|L}) + \frac{1}{2}x_k^T Q_k x_k. \tag{6.10}$$

where we have redefined the prediction horizon as $L := L - 1$ for simplicity of notation.

Using that $x_{k|L} = O_L x_k + H_L u_{k|L-1}$ and the plant model $x_{k+1} = Ax_k + Bu_k$ gives

$$\begin{aligned} x_{k+1|L} &= O_L A x_k + F_L^d u_{k|L} && (6.11)\\ &= p_L + F_L^d u_{k|L}, && (6.12)\end{aligned}$$

where

$$\begin{aligned} F_L^d &= \begin{bmatrix} O_L B & H_L^d \end{bmatrix} \in \mathbb{R}^{Ln \times Lr}, && (6.13)\\ p_L &= O_L A x_k. && (6.14)\end{aligned}$$

Here O_L is the extended observability matrix for the matrix pair $(D = I_{n\times n}, A)$ and H_L^d the Toeplitz matrix of the impulse response matrices $E = 0$, DB, DAB, $\ldots$, $DA^{L-2}B$ (also with $D = I_{n\ timesn}$).

With these definitions we write the objective eq. (6.10) in terms of the unknown controls $u_{k|L}$ as

$$J_k = \frac{1}{2}(u_{k|L}^T H u_{k|L} + 2f_L^T u_{k|L} + J_0) + \frac{1}{2}x_k^T Q_k x_k, \tag{6.15}$$

where

$$H = P_{k|L} + F_L^{dT} Q_{k|L} F_L^d, \tag{6.16}$$

$$f_L^T = p_L^T Q_{k|L} F_L^d, \tag{6.17}$$

$$J_0 = p_L^T Q_{k|L} p_L \tag{6.18}$$

where $F_L^{dT} = (F_L^d)^T$

6.3 Optimal control and minimum objective

The optimal control $u_{k|L}^*$ minimizing the objective eq. (6.15) is given by

$$\frac{\partial J_k}{\partial u_{k|L}} = \frac{1}{2}(2Hu_{k|L} + 2f_L) = 0 \Rightarrow u_{k|L}^* = -H^{-1} f_L, \tag{6.19}$$

where we have to ensure

$$\frac{\partial^2 J_k}{\partial u_{k|L}^2} = H > 0, \tag{6.20}$$

for a minimum.

The minimum of the objective function is then

$$J_k^* = \frac{1}{2}(f_L^T H^{-1} f_L - 2 f_L^T H^{-1} f_L + J_0) + \frac{1}{2} x_k^T Q_k x_k, \tag{6.21}$$

$$= \frac{1}{2}(-f_L^T H^{-1} f_L + J_0) + \frac{1}{2} x_k^T Q_k x_k \tag{6.22}$$

$$= \frac{1}{2}(p_L^T (Q_{k|L} - Q_{k|L} F_L^d H^{-1} F_L^{dT} Q_{k|L}) p_L + \frac{1}{2} x_k^T Q_k x_k. \tag{6.23}$$

The minimum of the objective can then be written as a function of the present state x_k and the solution of the Riccati equation R_k as

$$J_k^* = \frac{1}{2} x_k^T R_k x_k, \tag{6.24}$$

where the solution to the Riccati equation is given by

$$R_k = (O_L A)^T (Q_{k|L} - Q_{k|L} F_L^d H^{-1} F_L^{dT} Q_{k|L}) O_L A + Q_k. \tag{6.25}$$

Interestingly, using eq. (6.16) we write eq. (6.25) as

$$R_k = (O_L A)^T (Q_{k|L} - Q_{k|L} F_L^d (P_{k|L} + F_L^{dT} Q_{k|L} F_L^d)^{-1} F_L^{dT} Q_{k|L}) O_L A + Q_k, \tag{6.26}$$

$$Q_{L|L} = S_N, \tag{6.27}$$

and comparing with the Riccati difference equation, we find strong similarities. The Riccati difference equation (6.4) may be replaced with a analytic matrix equation as in (6.26). The Riccati difference equation and the matrix eq. (6.26) are dual equations, meaning that if we replace the matrices in (6.4) with $A := O_L A$, $B := F_L^d$,

$R_{k+1} := Q_{k|L}$ and $P_k := P_{k|L}$ we obtain the analytic matrix expression (6.26) for R_k.

Notice that the Riccati matrix R_k is equal to the steady state solution R of the Discrete Algebraic Riccati Equation (DARE) when the prediction horizon L is large. However, if the last lower left block in $Q_{k|L}$, is chosen equal to R then $R_k = R$ for any finite prediction horizon $1 < L$, and hence the optimal control (6.19) is stabilizing.

Let us now study the future controlled responses. We write the predicted control actions as follows

$$u^*_{k|L} = -H^{-1} f_L = -H^{-1} F_L^{dT} Q_{k|L} p_L = G_L x_k, \tag{6.28}$$

where we have defined the gain matrix

$$G_L = -H^{-1} F_L^{dT} Q_{k|L} O_L A. \tag{6.29}$$

Substituting the optimal control into (6.12) gives the predicted controlled state responses

$$\begin{aligned} x_{k+1|L} &= O_L A x_k + F_L^d u^*_{k|L} \\ &= (O_L A + F_L^d G_L) x_k \end{aligned} \tag{6.30}$$

From this we may also deduce the following alternative formulation of the Riccati matrix

$$R_k = G_L^T H G_L + 2(O_L A)^T Q_{k|L} F_L^d G_L + (O_L A)^T Q_{k|L} O_L A + Q_k. \tag{6.31}$$

Chapter 7

Discrete LQ optimal control: Alternative direct solution

7.1 Innledning

Vi har i avsnitt 5.2 vist at løsningen av det diskrete optimal reguleringsproblemet består av en Riccati-ligning. Dette betyr at for å finne de optimale pådrag må vi løse den diskrete Riccati-ligningen.

Vi skal i dette avsnittet vise at man ikke trenger å løse den diskrete Riccati ligningen som vist i avsnitt 5.2 for å finne den optimale løsningen. Dette resultatet er meget viktig fordi det blant annet viser sammenheng mellom klassisk LQ/LQG regulering og modell prediktiv regulering (MPC). I denne sammenheng er det av interesse å diskutere det diskrete LQ kriteriet.

7.2 Diskusjon av det diskrete LQ kriteriet

Dersom man skal sammenligne klassisk LQ regulering og såkallt modell prediktiv regulering er det en god ide å starte med å se på optimal kriteriet som benyttes.

La oss studere det diskrete optimal kriteriet

$$J_i = \frac{1}{2} x_N^T S_N x_N + \frac{1}{2} \sum_{k=i}^{N-1} (x_k^T Q_k x_k + u_k^T P_k u_k), \tag{7.1}$$

der S_N, Q_k og P_k er symmetriske vekt-matriser. i er det diskrete start-tidspunktet og N er det diskrete slutt-tidspunktet.

Vi tar utganhspunkt i denne formuleringen av et LQ kriterium fordi det er en forholdsvis generell formulering. Kriteriet (7.1) er dessuten identisk med det som benyttes i Lewis og Syrmos (1995) og Söderström (1994). Dette kan refereres til som det klassiske diskrete LQ kriteriet.

Dersom start-tidspunktet er $k = i$ og slutt-tidspunktet er $k = N$ vil kriteriet J_i være avhengig av pådragsvektoren ved $N-i$ diskrete tidspunkt, dvs., $u_k \; \forall \; k = i, \cdots, N-i$.

Vi forutsetter at $N > i$. Kriteriet er imidlertid avhengig av tilstandsvektoren x_k ved $N - i + 1$ diskrete tidspunkter. Merk også at det ikke er mulig å påvirke tilstanden x_i ved hjelp av noen av pådragene som inngår i kriteriet. Grunnen til dette er at pådraget u_k bare påvirker tilstanden ved neste tidspunkt, dvs., x_{k+1}. LQ kriteriet (7.1) kan derfor splittes opp i en sum av to deler. En del som er avhengig av pådragssekvensen og en del som er uavhengig av pådragene. Vi har

$$J_i = \frac{1}{2}\sum_{k=i}^{N-1}(x_{k+1}^T Q_{k+1} x_{k+1} + u_k^T P_k u_k) + \frac{1}{2}x_i^T Q_i x_i \tag{7.2}$$

Denne formuleringen av LQ kriteriet er identisk med (7.1) dersom $Q_N = S_N$. Det er klart at det bare er det første leddet på høyre side som kan påvirkes av pådragene.

LQ kriteriene (7.1) og (7.2) er videre definert over en tidshorisont på $L = N - i + 1$ diskrete tidspunkt. Merk også at dersom tidshorisonten L og start-tidspunkt i er gitt må vi ha at slutt-tidspunktet er gitt ved $N = L - 1 + i$. Setter vi dette inn i LQ kriteriet (7.1) får vi

$$J_i = \frac{1}{2}x_{L-1+i}^T S_{L-1+i} x_{L-1+i} + \frac{1}{2}\sum_{k=i}^{L-1+i-1}(x_k^T Q_k x_k + u_k^T P_k u_k) \tag{7.3}$$

Dette kriteriet er vel definert dersom horisonten L og initial-tidspunktet i er spesifiserte. Setter vi $N = L - 1 + i$ inn i LQ kriteriet (7.2) får vi

$$J_i = \frac{1}{2}\sum_{k=i}^{L-1+i-1}(x_{k+1}^T Q_{k+1} x_{k+1} + u_k^T P_k u_k) + \frac{1}{2}x_i^T Q_i x_i \tag{7.4}$$

Dersom $L > 0$ er en konstant vil den første delen av kriteriet være definert over en konstant horisont på $L - 1$ diskrete tidspunkter uavhengig av initial-tidspunktet i.

Merknad 7.1 *De fire formuleringene av LQ kriteriet gitt ved (7.1), (7.2), (7.3) og (7.4) er identiske dersom initial-tidspunktet i og slutt-tidspunktet N er spesifisert. Vi forutsetter at $Q_N = S_N$ i ligning (7.2) og at $Q_N = S_N$ og $L = N - i + 1$ i (7.4).*

Merknad 7.2 *Formuleringen i (7.2) viser at vi kan separere ut kvadrat formen $\frac{1}{2}x_i^T Q_i x_i$ fra det klassiske LQ kriteriet (7.1). Ingen av pådragene som inngår i kriteriet har innvirkning på denne kvadrat-formen. Vi forutsetter her at systemet er strengt-proper. Dette betyr at det er mulig å finne de optimale pådragene ved å finne minimum av det første leddet på venstre side av (7.2).*

Et viktig spesialtilfelle får vi nå ved å velge i lik løpende tid. Formuleringene gitt ved (7.3) og (7.4) av LQ kriteriet (7.1) refereres da til som *receding horizon* LQ kriterier. Et LQ kriterium av denne typen gir opphav til et nytt optimaliseringsproblem for hvert nytt tidspunkt.

Vi merker oss i denne sammenheng at LQ kriteriet (7.3) kan skrives som

$$\begin{aligned} J_k &= \tfrac{1}{2}x_{k+L-1}^T S_{k+L-1} x_{k+L-1} \\ &\tfrac{1}{2}\textstyle\sum_{i=1}^{L-1}(x_{k+i-1}^T Q_{k+i-1} x_{k+i-1} + u_{k+i-1}^T P_{k+i-1} u_{k+i-1}) \end{aligned} \tag{7.5}$$

der k er løpende diskret tid. På samme måte kan LQ kriteriet (7.4) skrives slik

$$J_k = \frac{1}{2}\sum_{i=1}^{L-1}(x_{k+i}^T Q_{k+i} x_{k+i} + u_{k+i-1}^T P_{k+i-1} u_{k+i-1}) + \frac{1}{2} x_k^T Q_k x_k \tag{7.6}$$

LQ kriterier som vist i de to siste ligningene benyttes i stor grad i forbindelse med MPC.

Merknad 7.3 (receding-horizon control) *Dersom vi for hvert nytt diskrete tidspunkt k minimaliserer et LQ kriterium av typen (7.5) eller(7.6) med hensyn til pådragene u_k, $u_{k+1}, \ldots, u_{k+L-2}$ og bare benytter det første pådraget u_k til å regulere prosessen så refereres dette til som* receding-horizon control. *Dette refereres i litteraturen også til som Model Predictive Control (MPC) og Moving Horizon Control (MHC).*

7.3 Diskret optimal regulering: Alternativ løsning I

La oss fortsette diskusjonen med to eksempler.

Example 7.1 (kompakt formulering av optimal kriteriet)
Gitt en tidshorisont $L = 4$. Vi spesifiserer start-tidspunktet til $i = 0$. Dette betyr at slutt-tiden er $N = L + i - 1 = 3$. Vi definerer et kvadratisk optimal kriterium over tidshorisonten

$$J_0 = \frac{1}{2} x_3^T S_3 x_3 + \frac{1}{2}\sum_{k=0}^{2}(x_k^T Q_k x_k + u_k^T P_k u_k) \tag{7.7}$$

Poenget med dette eksemplet er å vise at kriteriet kan skrives på matriseform. Vi har

$$J_0 = \frac{1}{2}(x_{0|4}^T Q_{0|4} x_{0|4} + u_{0|3}^T P_{0|3} u_{0|3}) \tag{7.8}$$

der

$$x_{0|4} = \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad u_{0|3} = \begin{bmatrix} u_0 \\ u_1 \\ u_2 \end{bmatrix}. \tag{7.9}$$

og

$$Q_{0|4} = \begin{bmatrix} Q_0 & 0 & 0 & 0 \\ 0 & Q_1 & 0 & 0 \\ 0 & 0 & Q_2 & 0 \\ 0 & 0 & 0 & S_3 \end{bmatrix}, \quad P_{0|3} = \begin{bmatrix} P_0 & 0 & 0 \\ 0 & P_1 & 0 \\ 0 & 0 & P_2 \end{bmatrix}. \tag{7.10}$$

Det er en fin øving å vise dette !
△

Example 7.2 (Utvidet tilstandsrommodell)
La oss studere kriteriet (7.8). Det er av interesse å uttrykke $x_{0|4}$ ved hjelp av pådragsvektoren $u_{0|3}$. Kriteriet kan da uttrykkes som en funksjon av $u_{0|3}$. Vi kan dermed finne den optimale pådragsvektoren $u_{0|3}$ ved å sette den deriverte av kriteriet mht. pådragsvektoren lik null.

Vi skal nå vise at en slik sammenheng eksisterer. Med utgangspunkt i tilstandsrommodellen finner vi

$$\overbrace{\begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix}}^{x_{0|4}} = \overbrace{\begin{bmatrix} I \\ A \\ A^2 \\ A^3 \end{bmatrix}}^{O_4} x_0 + \overbrace{\begin{bmatrix} 0 & 0 & 0 \\ B & 0 & 0 \\ AB & B & 0 \\ A^2B & AB & B \end{bmatrix}}^{H_4^2} \overbrace{\begin{bmatrix} u_0 \\ u_1 \\ u_2 \end{bmatrix}}^{u_{0|3}}. \tag{7.11}$$

Vi har funnet sammenhengen

$$x_{0|4} = O_4 x_0 + H_4^d u_{0|3}. \tag{7.12}$$

Merk at matrisen O_4 er en (utvidet) observerbarhetsmatrise for matriseparet (D, A) der $D = I$.

Dersom vi setter sammenhengen (7.12) inn i kriteriet så vil kriteriet bare avhenge av $u_{0|3}$ og x_0. x_0 er uavhengig av $u_{0|3}$. Den optimale pådragsvektoren kan dermed finnes ved å sette den deriverte av J_0 med hensyn til $u_{0|3}$ lik null.
$\triangle$

Det kan vises at det diskrete optimal kriteriet (7.1) generelt kan skrives på den kompakte matriseformen

$$J_i = \frac{1}{2}(x_{i|L}^T Q_{i|L} x_{i|L} + u_{i|L-1}^T P_{i|L-1} u_{i|L-1}), \tag{7.13}$$

der

$$x_{i|L} = \begin{bmatrix} x_i \\ x_{i+1} \\ \vdots \\ x_{L+i-2} \\ x_{L+i-1} \end{bmatrix}, \quad u_{i|L-1} = \begin{bmatrix} u_i \\ u_{i+1} \\ \vdots \\ u_{L+i-2} \end{bmatrix}. \tag{7.14}$$

Videre har vi den utvidede tilstandsrommodellen

$$x_{i|L} = O_L x_i + H_L^d u_{i|L-1}. \tag{7.15}$$

Setter vi (7.15) inn i kriteriet (7.13) får vi at

$$J_i = \frac{1}{2}[(O_L x_i + H_L^d u_{i|L-1})^T Q_{i|L}(O_L x_i + H_L^d u_{i|L-1}) + u_{i|L-1}^T P_{i|L-1} u_{i|L-1}]. \tag{7.16}$$

Vi finner

$$\frac{\partial J_i}{\partial u_{i|L-1}} = H_L^{dT} Q_{i|L} O_L x_i + (H_L^{dT} Q_{i|L} H_L^d + P_{i|L-1}) u_{i|L-1}. \tag{7.17}$$

Vi setter den deriverte lik null og finner følgende uttrykk for den optimale pådragsvektoren

$$u_{i|L-1} = Gp_L, \tag{7.18}$$

der

$$G = -(H_L^{dT} Q_{i|L} H_L^d + P_{i|L-1})^{-1} H_L^{dT} Q_{i|L}, \tag{7.19}$$

$$p_L = O_L x_i. \tag{7.20}$$

Legg merke til at p_L representerer tilstandsrommodellens autonome responser ved de diskrete tidspunktene $i, i+1, i+2, \cdots, L+i-1$. Dvs. p_L inneholder løsningene av tilstandsrommodellen $x_{k+1} = Ax_k$ der initial tilstandsvektoren x_i er gitt. Første blokk i p_L er identisk med x_i. Det kan vises at første blokk kolonne i G_L er lik null. Grunnen til dette er at de optimale pådragene i $u_{i|L-1}$ ikke er avhengig av initial-tilstandsvektoren x_i. Vi skal i neste avsnitt vise hvordan vi, ved å ta hensyn til dette, kan utlede en alternativ formulering av resultatet i (7.18)-(7.20)

Minimumsverdien av kriteriet blir

$$J_i^* = \frac{1}{2} x_i^T O_L^T [(I + H_L^d G)^T Q_{i|L} (I + H_L^d G) + G^T P_{i|L-1} G] O_L x_i. \tag{7.21}$$

Sammenligner vi dette med den løsningen som er presentert i avsnitt 5.2 så finner vi følgende uttrykk for løsningen av den diskrete Riccati-ligningen ved start-tidspunktet $k = i$.

$$R_i = O_L^T [(I + H_L^d G)^T Q_{i|L} (I + H_L^d G) + G^T P_{i|L-1} G] O_L. \tag{7.22}$$

Dersom tidshorisonten L er stor vil (7.22) konvergere mot den stasjonære løsningen av den diskrete algebraiske Riccati-ligningen. Dette betyr at vi har funnet en alternativ løsningsmetode for det diskrete LQ optimal reguleringsproblemet. Vi har dessuten funnet en alternativ metode for å løse den diskrete Riccati-ligningen på .

La oss studere det lukkede systemets responser. Vi setter det optimale pådraget gitt ved (7.18)-(7.20) inn i den utvidede tilstandsrom-modellen (7.15) og får

$$x_{i|L} = (I + F_L^d G) O_L x_i. \tag{7.23}$$

Det er klart at denne ligningen kan benyttes til å studere stabilitets egenskapene til det lukkede systemet.

Stabilitet er ikke nødvendigvis relevant i forbindelse med optimaliseringsproblemer der vi benytter et endelig optimaliseringsintervall $i \leq k \leq N$. I batch prosess reguleringsproblemer og minimum-tid reguleringsproblemer er det normalt ikke nødvendig å kreve stabilitet, dvs. en analyse av systemet når $t \to \infty$ har ingen mening. Dersom vi imidlertid krever stabilitet vil det i enkelte tilfeller være lurt å vektlegge slutt-tilstanden.

7.4 Diskret optimal regulering: Alternativ løsning II

Vi skal i dette avsnittet vise at resultatene som ble funnet i avsnitt 5.2 kan uttrykkes på en noe enklere måte. Det kan vises at det diskrete optimal kriteriet (7.13) kan splittes i to deler.

$$J_i = \frac{1}{2}(x_{i+1|L-1}^T Q_{i+1|L-1} x_{i+1|L-1} + u_{i|L-1}^T P_{i|L-1} u_{i|L-1}) + \frac{1}{2} x_i^T Q_i x_i. \quad (7.24)$$

Grunnen til at vi har splittet opp kriteriet er at den utvidede pådragsvektoren $u_{i|L-1}$ bare kan påvirke den utvidede tilstandsvektoren $x_{i+1|L-1}$. Grunnen til dette er at u_k bare påvirker tilstanden ved neste tidspunkt, dvs., x_{k+1}. Dvs., $u_{i|L-1}$ kan ikke påvirke tilstandsvektoren x_i ved start-tidspunktet.

Dersom vi tar utgangspunkt i formuleringen av kriteriet som gitt i (7.24) finner vi en annen formulering av løsningen en den som ble utledet i avsnitt (7.3). Løsningene er imidlertid identiske.

Tilsvarende (7.15) finner vi følgende formulering

$$x_{i+1|L-1} = O_{L-1} A x_i + F_{L-1}^d u_{i|L-1}, \quad (7.25)$$

der

$$F_{L-1}^d = \left[\, O_{L-1} B \;\; H_{L-1}^d \,\right] \;\in\; \mathbb{R}^{(L-1)n \times (L-1)r}. \quad (7.26)$$

Vi vil referere til (7.25) som en utvidet tilstandsrommodell. Vi setter nå ligning (7.25) inn i kriteriet (7.24) og får.

$$\begin{aligned} J_i = \tfrac{1}{2}[&(O_{L-1} A x_i + F_{L-1}^d u_{i|L-1})^T Q_{i+1|L-1} (O_{L-1} A x_i + F_{L-1}^d u_{i|L-1}) \\ &+ u_{i|L-1}^T P_{i|L-1} u_{i|L-1}] + \tfrac{1}{2} x_i^T Q_i x_i. \end{aligned} \quad (7.27)$$

Kriteriet kan skrives slik

$$\begin{aligned} J_i = &\tfrac{1}{2} u_{i|L-1}^T (P_{i|L-1} + F_{L-1}^{dT} Q_{i+1|L-1} F_{L-1}^d) u_{i|L-1} + (O_{L-1} A x_i)^T Q_{i+1|L-1} F_{L-1}^d u_{i|L-1} \\ &+ \tfrac{1}{2} x_i^T [(O_{L-1} A)^T Q_{i+1|L-1} O_{L-1} A + Q_i] x_i. \end{aligned} \quad (7.28)$$

Vi kan finne en betingelse for minimum ved å derivere J_i med hensyn på $u_{i|L-1}$. Derivasjon gir

$$\frac{\partial J_i}{\partial u_{i|L-1}} = F_{L-1}^{dT} Q_{i+1|L-1} O_{L-1} A x_i + (F_{L-1}^{dT} Q_{i+1|L-1} F_{L-1}^d + P_{i|L-1}) u_{i|L-1}. \quad (7.29)$$

Vi setter ligning (7.29) lik null og får

$$u_{i|L-1} = G_{L-1} p_{L-1}, \quad (7.30)$$

der vi definerer

$$G_{L-1} = -(F_{L-1}^{dT} Q_{i+1|L-1} F_{L-1}^d + P_{i|L-1})^{-1} F_{L-1}^{dT} Q_{i+1|L-1}, \quad (7.31)$$

$$p_{L-1} = O_{L-1} A x_i. \quad (7.32)$$

Legg merke til at p_{L-1} inneholder det åpne systemets autonome responser ved tidspunktene $i+1, i+2, \cdots, L+i-1$. Vi har her utledet en litt annen formulering en den presentert i (7.18)-(7.20).

For at løsningen (7.30)-(7.32) garantert skal være den optimale løsningen som gir minimum av kriteriet ma den Hessiske matrisen være positiv definit. Dvs., vi har følgende krav

$$\frac{\partial^2 J_i}{\partial u_{i|L-1}^2} = (F_{L-1}^{dT} Q_{i+1|L-1} F_{L-1}^d + P_{i|L-1}) > 0. \tag{7.33}$$

Dette vil alltid være oppfyllt dersom vi for eksempel velger $P_k > 0 \; \forall \; k = i, \cdots, L + i - 2$. Dvs. dersom vi velger positiv definite vektmatriser for pådragsvektoren ved alle diskrete tidspunkt.

La oss studere det lukkede systemets responser. Vi setter det optimale pådraget gitt ved (7.30)-(7.32) inn i den utvidede tilstandsrom-modellen (7.25) og får

$$x_{i+1|L-1} = (O_{L-1}A + F_{L-1}^d G_{L-1} O_{L-1} A) x_i. \tag{7.34}$$

Det er klart at denne ligningen kan benyttes til å studere stabilitets egenskapene til det lukkede systemet.

La oss finne minimumsverdien til kriteriet. Vi setter den optimale pådragsvektoren (7.30) inn i kriteriet (7.27) og finner

$$\begin{aligned} J_i^* = {} & \tfrac{1}{2} x_i^T (O_{L-1}A)^T [(I + F_{L-1}^d G_{L-1})^T Q_{i+1|L-1} (I + F_{L-1}^d G_{L-1}) \\ & + G_{L-1}^T P_{i|L-1} G_{L-1}] O_{L-1} A x_i + \tfrac{1}{2} x_i^T Q_i x_i \quad . \end{aligned} \tag{7.35}$$

Med utgangspunkt i maksimumsprinsippet kan vi vise at minimumsverdien av kriteriet er gitt ved $J_i^* = \frac{1}{2} x_i^T R_i x_i$ der R_i er løsning av den diskrete Riccati-ligningen. Dette betyr at løsningen av den diskrete Riccati-ligningen, ved tiden i, er gitt ved

$$\begin{aligned} R_i = {} & (O_{L-1}A)^T [(I + F_{L-1}^d G_{L-1})^T Q_{i+1|L-1} (I + F_{L-1}^d G_{L-1}) + G_{L-1}^T P_{i|L-1} G_{L-1}] O_{L-1} A \\ & + Q_i. \end{aligned} \tag{7.36}$$

Dette resultatet er viktig fordi det viser at det finnes en "analytisk" løsning av den diskrete Riccati-ligningen.

En alternativ formulering finner vi ved å sette den optimale pådragsvektoren (7.30) inn i kriteriet (7.28). Dette gir

$$R_i = -Z^T \mathcal{H} Z + (O_{i+1|L-1}A)^T Q_{i+1|L-1} O_{i+1|L-1} A + Q_i, \tag{7.37}$$

der

$$Z = F_{L-1}^{dT} Q_{i+1|L-1} O_{L-1} A, \tag{7.38}$$

$$\mathcal{H} = (P_{i|L-1} + F_{L-1}^{dT} Q_{i+1|L-1} F_{L-1}^d)^{-1}. \tag{7.39}$$

Merknad 7.4 *Veklegging av slutt-tilstanden er viktig for stabilitet i forbindelse med endelig horisont LQ regulering.*

Dersom den stasjonære løsningen av Riccati ligningen skal finnes ved hjelp av formlene gitt over kan det være hensiktsmessig med en tilstrekkelig vekting av slutt-tilstanden. Slutt-tilstanden vektlegges med matrisen S_N

Merk at dersom vi vekter slutt-tilstanden med $S_N = R$ *der* R *er den stasjonære løsningen av Riccati ligningen vil det lukkede systemet være stabilt selv om vi velger en endelig horisont på kriteriet. Se også oppgave* **??**.

Chapter 8

Time delay in optimal systems

8.1 Modeling of time delay

We will in this section discuss systems with possibly time delay. Assume that the system without time delay is given by a proper state space model as follows

$$x_{k+1} = Ax_k + Bu_k, \tag{8.1}$$

$$y_k^- = Dx_k + Eu_k, \tag{8.2}$$

and that the output of the system, y_k, is identical to, y_k^-, but delayed a delay τ samples. The time delay may then be exact expressed as

$$y_{k+\tau} = y_k^-. \tag{8.3}$$

Discrete time systems with time delay may be modeled by including a number of fictive dummy states for describing the time delay. Some alternative methods are described in the following.

8.1.1 Transport delay and controllability canonical form

Formulation 1: State space model for time delay

We will include a positive integer number τ fictive dummy state vectors of dimension m in order for describing the time delay, i.e.,

$$\left.\begin{array}{rcl} x_{k+1}^1 &=& Dx_k + Eu_k \\ x_{k+1}^2 &=& x_k^1 \\ &\vdots& \\ x_{k+1}^\tau &=& x_k^{\tau-1} \end{array}\right\} \tag{8.4}$$

The output of the process is then given by

$$y_k = x_k^\tau \tag{8.5}$$

We se by comparing the defined equations (8.4) and (8.5) is an identical description as the original state space model given by (8.1), (8.2 and (8.3). Note that we in

(8.4) have defined a number τm fictive dummy state variables for describing the time delay.

Augmenting the model (8.1) and (8.2) with the state space model for the delay gives a complete model for the system with delay.

$$\overbrace{\begin{bmatrix} x \\ x^1 \\ x^2 \\ \vdots \\ x^\tau \end{bmatrix}_{k+1}}^{\tilde{x}_{k+1}} = \overbrace{\begin{bmatrix} A & 0 & 0 & \cdots & 0 & 0 \\ D & 0 & 0 & \cdots & 0 & 0 \\ 0 & I & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & I & 0 \end{bmatrix}}^{\tilde{A}} \overbrace{\begin{bmatrix} x \\ x^1 \\ x^2 \\ \vdots \\ x^\tau \end{bmatrix}_k}^{\tilde{x}_k} + \overbrace{\begin{bmatrix} B \\ E \\ 0 \\ \vdots \\ 0 \end{bmatrix}}^{\tilde{B}} u_k \tag{8.6}$$

$$y_k = \overbrace{\begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & I \end{bmatrix}}^{\tilde{D}} \overbrace{\begin{bmatrix} x \\ x^1 \\ x^2 \\ \vdots \\ x^{\tau-1} \\ x^\tau \end{bmatrix}_k}^{\tilde{x}_k} \tag{8.7}$$

hence we have

$$\tilde{x}_{k+1} = \tilde{A}\tilde{x}_k + \tilde{B}u_k \tag{8.8}$$
$$y_k = \tilde{D}\tilde{x}_k \tag{8.9}$$

where the state vector $\tilde{x}_k \in \mathbb{R}^{n+\tau m}$ contains n states for the process (8.1) without delay and a number τm states for describing the time delay (8.3).

With the basis in this state space model, Equations (8.8) and (8.9), we may use all our theory for analyse and design of linear dynamic control systems.

Formulation 2: State space model for time delay

The formulation of the time delay in Equations (8.6) and (8.7) is not very compacter. We will in this section present a different more compact formulation. In some circumstances the model from y_k^- to y_k will be of interest in itself. We start by isolating this model. Consider the following state space model where $y_k- \in \mathbb{R}^m$ s delayed an integer number τ time instants.

$$\overbrace{\begin{bmatrix} x^1 \\ x^2 \\ x^3 \\ \vdots \\ x^\tau \end{bmatrix}_{k+1}}^{x^\tau_{k+1}} = \overbrace{\begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ I & 0 & 0 & \cdots & 0 & 0 \\ 0 & I & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & I & 0 \end{bmatrix}}^{A^\tau} \overbrace{\begin{bmatrix} x^1 \\ x^2 \\ x^3 \\ \vdots \\ x^\tau \end{bmatrix}_k}^{x^\tau_k} + \overbrace{\begin{bmatrix} I \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}}^{B^\tau} y_k^- \tag{8.10}$$

$$y_k = \overbrace{\begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & I \end{bmatrix}}^{D^\tau} \overbrace{\begin{bmatrix} x \\ x^1 \\ x^2 \\ \vdots \\ x^{\tau-1} \\ x^\tau \end{bmatrix}_k}^{x_k^\tau} \tag{8.11}$$

which may be written as

$$x_{k+1}^\tau = A^\tau x_k^\tau + B^\tau y_k^- \tag{8.12}$$

$$y_k = D^\tau x_k^\tau \tag{8.13}$$

where $x_k^\tau \in \mathbb{R}^{\tau m}$. the initial state for the delay state is put to $x_0^\tau = 0$. Note here that the state space model (8.10) and (8.11) is on so called controllability canonical form.

Combining (8.12) and (8.13) with the state space model equations (8.1) and (8.2), gives an compact model for the entire system, i.e., the system without delay from u_k to y_k^-, and for the delay from y_k^- to the output y_k.

$$\overbrace{\begin{bmatrix} x \\ x^\tau \end{bmatrix}_{k+1}}^{\tilde{x}_k} = \overbrace{\begin{bmatrix} A & 0 \\ B^\tau D & A^\tau \end{bmatrix}}^{\tilde{A}} \overbrace{\begin{bmatrix} x \\ x^\tau \end{bmatrix}_k}^{\tilde{x}_k} + \overbrace{\begin{bmatrix} B \\ B^\tau E \end{bmatrix}}^{\tilde{B}} u_k \tag{8.14}$$

$$y_k = \overbrace{\begin{bmatrix} 0 & D^\tau \end{bmatrix}}^{\tilde{D}} \overbrace{\begin{bmatrix} x \\ x^\tau \end{bmatrix}_k}^{\tilde{x}_k} \tag{8.15}$$

Note that the state space model given by Equations (8.14) and (8.15), is identical with the state space model in (8.6) and (8.7).

8.1.2 Time delay and observability canonical form

A simple method for modeling the time delay may be obtained by directly taking Equation (8.3) as the starting point. Combining $y_{k+\tau} = y_k^-$ with a number $\tau - 1$ fictive dummy states, $y_{k+1} = y_{k+1}$, $\cdots$, $y_{k+\tau-1} = y_{k+\tau-1}$ we may write down the following state space model

$$\overbrace{\begin{bmatrix} y_{k+1} \\ y_{k+2} \\ y_{k+3} \\ \vdots \\ y_{k+\tau} \end{bmatrix}}^{x_{k+1}^\tau} = \overbrace{\begin{bmatrix} 0 & I & 0 & \cdots & 0 & 0 \\ 0 & 0 & I & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & I \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}}^{A^\tau} \overbrace{\begin{bmatrix} y_k \\ y_{k+1} \\ y_{k+2} \\ \vdots \\ y_{k+\tau-1} \end{bmatrix}}^{x_k^\tau} + \overbrace{\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ I \end{bmatrix}}^{B^\tau} y_k^- \tag{8.16}$$

$$y_k = \overbrace{\begin{bmatrix} I & 0 & 0 & \cdots & 0 \end{bmatrix}}^{D^\tau} \overbrace{\begin{bmatrix} y_k \\ y_{k+1} \\ y_{k+2} \\ \vdots \\ y_{k+\tau-1} \end{bmatrix}}^{x_k^\tau} \tag{8.17}$$

where $x_k^\tau \in \mathbb{R}^{\tau m}$.

The initial state for the time delay is put to $x_0^\tau = 0$. Note that the state space model (8.16) and (8.17) is on observability canonical form.

8.2 Implementation of time delay

The state space model for the delay contains a huge number of zeroes and ones when the time delay is large, ie when the delay state space model dimension $m\tau$ is large.

In the continuous time we have that a delay is described exact by $y_{k+\tau} = y_k^-$. It can be shown that instead of simulating the state space model for the delay we can obtain the same by using a matrix (array or shift register) of size $n_\tau \times m$ where we use $n_\tau = \tau$ as an integer number of delay samples.

The above state space model for the delay contains $n_\tau = \tau$ state equations which may be expressed as

$$\begin{aligned} x_k^1 &= y_{k-1}^- \\ x_k^2 &= x_{k-1}^1 \\ &\vdots \\ x_k^{\tau-1} &= x_{k-1}^{\tau-2} \\ y_k &= x_{k-1}^{\tau-1} \end{aligned} \tag{8.18}$$

where we have used $y_k = x_k^\tau$. This may be implemented efficiently by using a matrix (or vector x. The following algorithm (or variants of it) may be used:

Algorithm 8.2.1 (Implementing time delay of a signal)
Given a vector $y_k^- \in \mathbb{R}^m$. A time delay of the elements in the vector y_k^- of n_τ time instants (samples) may simply be implemented by using a matrix x of size $n_\tau \times m$.

At each sample, k, (each call of the algorithm) do the following:

1. *Put y_k^- in the first row (at the top) of the matrix x.*
2. *Interchange each row (elements) in matrix one position down in the matrix.*
3. *The delayed output y_k is taken from the bottom element (last row) in the matrix x.*

$y_k = x(\tau, 1:m)^T$
for $i = \tau : -1 : 2$
$\quad x(i, 1:m) = x(i-1, 1:m)$
end
$x(1, 1:m) = (y_k^-)^T$

Note that this algorithm should be evaluated at each time instant k.
$\triangle$

8.3 Optimal regulering av systemer med transportforsinkelse

8.3.1 Løsning ved å modellere transportforsinkelsen

Transisjonsmatrisen $\tilde{A}$ til systemet med transportforsinkelse er singulær. Grunnen til dette er at transportforsinkelsesmodellen inkluderer τm egenverdier i origo. Dersom vi studerer den optimale løsningen vil vi også finne at transisjonsmatrisen til det lukkede systemet er singulær. Vi skal se at det lukkede systemet uansett valg av vektmatriser vil ha m egenverdier i origo.

Den optimale tilbakekoplingsmatrisen er gitt av

$$G_k = -(P + \tilde{B}^T R_{k+1} \tilde{B})^{-1} \tilde{B}^T R_{k+1} \tilde{A} \tag{8.19}$$

Dersom vi benytter formuleringen (8.6 kan vi uttrykke $\tilde{A}$ som

$$\tilde{A} = \begin{bmatrix} A_{11} & 0_{n+(\tau-1)m \times m} \\ A_{12} & 0_{m \times m} \end{bmatrix} \tag{8.20}$$

Vi får dermed at G_k beregnes som

$$G_k = \overbrace{\begin{bmatrix} \times & \times \end{bmatrix}}^{-(P+\tilde{B}^T R_{k+1} \tilde{B})^{-1} \tilde{B}^T R_{k+1}} \overbrace{\begin{bmatrix} A_{11} & 0_{n+(\tau-1)m \times m} \\ A_{12} & 0_{m \times m} \end{bmatrix}}^{\tilde{A}} = \begin{bmatrix} G_1 & 0_{m \times m} \end{bmatrix} \tag{8.21}$$

der $G_1 \in \mathbb{R}^{r \times n+(\tau-1)}$ og der $\times$ betyr at denne matriseblokken generelt er forskjellig fra null.

Dette betyr at den optimale løsningen består av en tilbakekopling fra tilstandsvektoren x_k samt en tilbakekopling fra de $(\tau - 1)m$ første kunstige tilstandene som beskriver transportforsinkelsen. De siste m tilstandene i den kunstige tilstandsvektoren x_k^τ benytes altså ikke til å beregne den optimale tilbakekoplingen. Vi har fra definisjonen at $x_k^\tau = y_k$. Dette betyr at det optimale pådraget u_k er direkte uavhengig av y_k.

8.3.2 Løsning ved å modifisere LQ kriteriet

Anta en prosess

$$x_{k+1} = Ax_k + Bu_k \tag{8.22}$$

der utgangen til systemet er forsinket et helt antall τ sampler slik at

$$y_{k+\tau} = Dx_k \tag{8.23}$$

Dersom vi ikke vektlegger tilstandene som beskriver transportforsinkelsene vil vi få et (modifisert) LQ kriterium av formen

$$J_k = \sum_{i=1}^{L}(y_{k+i-1+\tau}^T Q_{k+i-1}^1 y_{k+i-1+\tau} + u_{k+i-1}^T P_{k+i-1} u_{k+i-1}) \tag{8.24}$$

som også kan splittes opp i to deler slik

$$J_k = \sum_{i=1}^{L}(y_{k+i+\tau}^T Q_{k+i}^1 y_{k+i+\tau} + u_{k+i-1}^T P_{k+i-1} u_{k+i-1}) + y_{k+\tau}^T Q_k^1 y_{k+\tau} \tag{8.25}$$

Det er bare første ledd på høyre side av LQ kriteriet som påvirkes av pådragene over prediksjonshorisonten.

Det vil her være rimelig at dersom man ikke vektlegger tilstandene som beskriver transportforsinkelsen så vil det optimale pådraget være generert av $u_k = G_k x_k$.

8.4 Numeriske eksempler

Example 8.1 (Optimalt system med transportforsinkelse)
Gitt et diskret 1. ordens system

$$x_{k+1} = Ax_k + Bu_k \tag{8.26}$$

$$y_k^- = Dx_k + Eu_k \tag{8.27}$$

der $A = 0.9$, $B = 0.5$, $D = 1$ og $E = -1$. Vi antar at det er en transportforsinkelse på $\tau = 1$ sample før utgangen y_k er tilgjengelig, dvs.

$$y_{k+1} = y_k^- \tag{8.28}$$

Vi får følgende tilstandsrommodell for totalsystemet

$$\begin{bmatrix} x_{k+1} \\ y_{k+1} \end{bmatrix} = \overbrace{\begin{bmatrix} A & 0 \\ D & 0 \end{bmatrix}}^{\tilde{A}} \begin{bmatrix} x_k \\ y_k \end{bmatrix} + \overbrace{\begin{bmatrix} B \\ E \end{bmatrix}}^{\tilde{B}} u_k \tag{8.29}$$

$$y_k = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_k \\ y_k \end{bmatrix} \tag{8.30}$$

Legg merke til at modellen (8.27) er bare proper, dvs. modellen inneholder en direkte innvirkning fra u_k til y_{k^-} mens modellen (8.30) er strengt proper, dvs. ingen direkte innvirkning fra u_k til utgangen.

Vi velger følgende glidende LQ kriterium

$$J_k = \sum_{i=1}^{L}(\begin{bmatrix} x_{k+i-1} & y_{k+i-1} \end{bmatrix} \overbrace{\begin{bmatrix} Q & 0 \\ D & Q_1 \end{bmatrix}}^{\tilde{Q}} \begin{bmatrix} x_{k+i-1} \\ y_{k+i-1} \end{bmatrix} + Pu_{k+i-1}^2) \tag{8.31}$$

med følgende vekter og tidshorisont

$$Q = 10,\ Q_1 = 10,\ P = 1,\ L = 15. \tag{8.32}$$

Dette gir

$$G_k = \begin{bmatrix} -0.610 & 0 \end{bmatrix},\ R_k = \begin{bmatrix} 56.183 & 0 \\ 0 & 10 \end{bmatrix},\ u_k = -0.61x_k \tag{8.33}$$

Vi ser at det optimale pådraget bare beregnes på bakgrunn av tilstanden x_k *i systemet. Det kan vises at den optimale tilbakekoplingen er uavhengig av den kunstige tilstanden* $x_k^1 = y_k$ *for alle valg av vektmatriser* $\tilde{Q}$ *(vi forutsetter at* $A, \tilde{Q}$ *er detekterbar). Dette kan vi for eksempel se ved å multiplisere ut uttrykket* $G_k = -(P + \tilde{B}^T R_{k+1} \tilde{B})^{-1} \tilde{B}^T R_{k+1} \tilde{A}$.

Vi har i dette eksemplet løst Riccati-ligningen ved å benyttet ligningene (5.52) og (5.53) ved å iterere bakover i tid fra slutt-tiden. Legg merke til at R *og* G *er tidsinvariante og at de bare varierer med horisonten* L.

Dersom $L \to \infty$ *(eller* L *er stor) får vi at* R *er løsningen av den diskrete algebraiske Riccati-ligningen (DARE). Ligningene (5.52) og (5.53) kan med fordel benyttes til å løse DARE. Det er her viktig og merke seg at MATLAB Control System Toolbox funksjonen dlqr.m ikke virker på dette systemet, dvs. dlqr.m klarer ikke å løse DARE. Grunnen til dette er at* $\tilde{A}$ *er singulær for dette eksemplet. dlqr.m kan ikke benyttes på systemer der transisjonsmatrisen er singulær. dlqr.m kan modifiseres ved å benytte en generalisert egenverdimetode presentert i Pappas og Laub (1980).*

Chapter 9

Examples on continuous time LQ optimal control

9.1 Examples: continuous time LQ-optimal control

Example 9.1 (LQ controller for distillation column)
A distillation column with one stage and re-boiler and accumulator can be modeled as

$$\dot{x}_1 = \frac{1}{M_1}(L_2 x_2 - L_1 x_1 - V y_1), \tag{9.1}$$

$$\dot{x}_2 = \frac{1}{M_2}(R x_3 + F x_F + V y_1 - L_2 x_2 - V y_2), \tag{9.2}$$

$$\dot{x}_3 = \frac{1}{M_3}(V y_2 - V x_3), \tag{9.3}$$

where x_1 is the composition in the re-boiler, x_2 is the composition in the column and x_3 is the top-product composition. The flow-rate of bottom product, L_1, and the flow-rate from the column, L_2, are given by

$$L_1 = R + F - V, \tag{9.4}$$

$$L_2 = R + F, \tag{9.5}$$

where R is the reflux (control input), V is the steam flow-rate from the re-boiler (control input) and F is the feed flow-rate. x_F is the feed composition. M_1 is the liquid in the reboiler, M_2 is the liquid holdup in the column and M_3 is the liquid in the accumulator.

The composition in the steam from the re-boiler, y_1, and from the column, y_2, are given by

$$y_1 = \frac{\alpha x_1}{1 + (\alpha - 1)x_1}, \tag{9.6}$$

$$y_2 = \frac{\alpha x_2}{1 + (\alpha - 1)x_2}. \tag{9.7}$$

This gives a non-linear model of the form

$$\dot{x} = f(x, u, v), \tag{9.8}$$

i.e.,

$$\dot{x}_1 = \frac{1}{M_1}((R+F)x_2 - (R+F-V)x_1 - V\frac{\alpha x_1}{1+(\alpha-1)x_1}), \tag{9.9}$$

$$\dot{x}_2 = \frac{1}{M_2}(Rx_3 + Fx_F + V\frac{\alpha x_1}{1+(\alpha-1)x_1} - L_2x_2 - V\frac{\alpha x_2}{1+(\alpha-1)x_2}), \tag{9.10}$$

$$\dot{x}_3 = \frac{1}{M_3}(V\frac{\alpha x_2}{1+(\alpha-1)x_2} - Vx_3), \tag{9.11}$$

where the parameters in the model are $M_1 = 10$, $M_2 = 5$, $M_3 = 10$, *and the relative volatility* $\alpha = 22.4$. *The control input vector,* u, *and the disturbance vector,* v, *with nominal values,* u_s, *and* v_s *are defined as*

$$u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} R \\ V \end{bmatrix}, \; u_s = \begin{bmatrix} 2 \\ 2.5 \end{bmatrix}, \tag{9.12}$$

$$v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} F \\ x_F \end{bmatrix}, \; v_s = \begin{bmatrix} 1 \\ 0.5 \end{bmatrix}. \tag{9.13}$$

Solving for the steady state composition profile, i.e., solving $\dot{x_s} = f(x_s, u_s, v_s) = 0$ *gives*

$$x_s = \begin{bmatrix} x_1^s \\ x_2^s \\ x_3^s \end{bmatrix} = \begin{bmatrix} 0.0500 \\ 0.4591 \\ 0.9500 \end{bmatrix}. \tag{9.14}$$

Note that x_s *can be computed by using an ODE solver. A linearized model around the steady state vectors* x_s, u_s *and* v_s *is given by*

$$\dot{\Delta x} = A\Delta x + B\Delta u + C\Delta v, \tag{9.15}$$

where $\Delta x = x - x_s$, $\Delta u = u - u_s$, $\Delta v = v - v_s$ *and*

$$A = \begin{bmatrix} -\frac{L_1^s + V_s K_1^s}{M_1} & \frac{L_2^s}{M_1} & 0 \\ \frac{V_s K_1^s}{M_2} & -\frac{L_2^s + V_s K_2^s}{M_2} & \frac{R^s}{M_2} \\ 0 & \frac{V_s K_2^s}{M_3} & -\frac{V_s}{M_3} \end{bmatrix} = \begin{bmatrix} -1.3576 & 0.3 & 0 \\ 2.6151 & -0.6956 & 0.4 \\ 0 & 0.0478 & -0.25 \end{bmatrix}, \tag{9.16}$$

$$B = \begin{bmatrix} \frac{x_2^s - x_1^s}{M_1} & \frac{x_1^s - y_1^s}{M_1} \\ \frac{x_3^s - x_2^s}{M_2} & \frac{y_1^s - y_2^s}{M_2} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0.0409 & -0.0491 \\ 0.0982 & -0.0818 \\ 0 & 0 \end{bmatrix}, \tag{9.17}$$

$$C = \begin{bmatrix} \frac{x_2^s - x_1^s}{M_1} & 0 \\ \frac{x_3^s - x_2^s}{M_2} & \frac{F_s}{M_2} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0.0409 & 0 \\ 0.0082 & 0.2 \\ 0 & 0 \end{bmatrix}. \tag{9.18}$$

where the steady state variables are $R_s = 2$, $V_s = 2.5$, $F_s = 1$, $x_F = 0.5$, $L_1^s = R_s + F_s - V_s = 0.5$, $L_2^s = R_s + F_s = 3$, *and*

$$K_i^s = \frac{\alpha}{(1+(\alpha-1)x_i^s)^2}, \; i = 1, 2. \tag{9.19}$$

$$y_i^s = \frac{\alpha x_i^s}{1+(\alpha-1)x_i^s}, \; i = 1, 2. \tag{9.20}$$

An infinite time LQ-optimal controller with the following weighting matrices

$$Q = \begin{bmatrix} 1000 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1000 \end{bmatrix}, \quad P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{9.21}$$

are given by $\Delta u = G\Delta x$, *i.e.,*

$$u = G(x - x_s) + u_s, \tag{9.22}$$

where

$$G = \begin{bmatrix} -13.2839 & -1.9656 & -7.5059 \\ 15.0669 & 2.0020 & 6.3353 \end{bmatrix}. \tag{9.23}$$

Hence the control inputs vary around the offset u_s *and the feedback seeks to minimize the deviation* $x - x_s$. *This example is implemented in the MATLAB script-file* **main_fcol3.m**.

Example 9.2 (LQ controller with integral action for distillation column)

Consider the distillation column model in Example 9.1. We want to include integral action in the controller. A state space model for the controller integrator is

$$\dot{z} = r - y = r - Dx, \tag{9.24}$$

where r *is the reference signal. Augmenting this with the state space model* $\dot{\Delta x} = A\Delta x + B\Delta u$ *gives*

$$\overbrace{\begin{bmatrix} \dot{\Delta x} \\ \dot{z} \end{bmatrix}}^{\dot{\tilde{x}}} = \overbrace{\begin{bmatrix} A & 0_{n\times m} \\ -D & 0_{m\times m} \end{bmatrix}}^{\tilde{A}} \overbrace{\begin{bmatrix} \Delta x \\ z \end{bmatrix}}^{\tilde{x}} + \overbrace{\begin{bmatrix} B \\ 0_{m\times r} \end{bmatrix}}^{\tilde{B}} \Delta u + \begin{bmatrix} 0_{n\times m} \\ I_{m\times m} \end{bmatrix} r. \tag{9.25}$$

This gives

$$\tilde{A} = \begin{bmatrix} -1.3576 & 0.3000 & 0 & 0 & 0 \\ 2.6151 & -0.6956 & 0.4000 & 0 & 0 \\ 0 & 0.0478 & -0.2500 & 0 & 0 \\ -1.0000 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1.0000 & 0 & 0 \end{bmatrix}, \tag{9.26}$$

$$\tilde{B} = \begin{bmatrix} 0.0409 & -0.0491 \\ 0.0982 & -0.0818 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}. \tag{9.27}$$

A simple solution is then to assume $r = 0$ *in the LQ-controller design procedure. Consider an LQ-objective where both the process state deviations,* Δx, *and the controller states,* z, *are weighted. We have*

$$J = \frac{1}{2}\int_{t_0}^{\infty} (\Delta x^T Q \Delta x + z^T Q_2 z + \Delta u^T P \Delta u) dt = \frac{1}{2}\int_{t_0}^{\infty} (\tilde{x}^T \tilde{Q} \tilde{x} + \Delta u^T P \Delta u) dt, \tag{9.28}$$

where the weigting matrix $\tilde{Q}$ is given by

$$\tilde{Q} = \begin{bmatrix} Q & 0_{n\times m} \\ 0_{m\times n} & Q_2 \end{bmatrix} = \begin{bmatrix} 1000 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1000 & 0 & 0 \\ 0 & 0 & 0 & 500 & 0 \\ 0 & 0 & 0 & 0 & 500 \end{bmatrix}. \tag{9.29}$$

This means that only the bottom-product and top-product compositions are weighted, in addition to the controller states. The LQ-controller is then found from the solution of the ARE

$$\tilde{A}^T R + R\tilde{A} - R\tilde{B}P^{-1}\tilde{B}^T R + \tilde{Q} = 0, \tag{9.30}$$

which gives the feedback matrix

$$G = -P^{-1}\tilde{B}^T R. \tag{9.31}$$

This gives

$$G = \begin{bmatrix} G_1 & G_2 \end{bmatrix} = \begin{bmatrix} -13.9059 & -4.8668 & -76.7138 & 2.6790 & 22.1996 \\ 22.8848 & 1.4733 & -20.1423 & -22.1996 & 2.6790 \end{bmatrix}. \tag{9.32}$$

The final LQ-controller with integral action can be implemented as

$$u = G_1(x - x_s) + G_2 z + u_s, \tag{9.33}$$

The practical implementation can be illustrated by the following MATLAB code lines

```
x=xs;                       % Initial values for the process states.
z=[0;0];                    % Initial values for the controller states.
r=[0.05;0.96];              % Reference signal.
for i=1:N
   y=D*x;                   % Process measurements.
   u=G1*(x-xs)+G2*z+us;     % LQ-controller with integral action.
   z=z+h*(r-y);             % Update controller state.
   Y(i,:)=y'; U(i,:)=u';    % Store outputs and inputs.
   f=fcol3(t,x,u,vs);       % Putting control input to the process,
   x=x+h*f;                 % updating the process model.
end
```

The order of the computations is of central importance. This should be noted by the reader. All details of this example is implemented in the MATLAB script-file **main_fcol3.m**. *A simulation of the closed loop system is illustrated in Figure 9.1.*

Example 9.3 (Equivalence between LQ and PD controllers)
A single input and single output linear system $\dot{x} = Ax + Bu$ and $y = Dx$ can be transformed to a canonical observability form, provided the pair (D, A) is observable. Consider a 2nd order system on canonical observability form

$$A = \begin{bmatrix} 0 & 1 \\ a_0 & a_1 \end{bmatrix}, B = \begin{bmatrix} b_0 \\ b_1 \end{bmatrix}, D = \begin{bmatrix} 1 & 0 \end{bmatrix}. \tag{9.34}$$

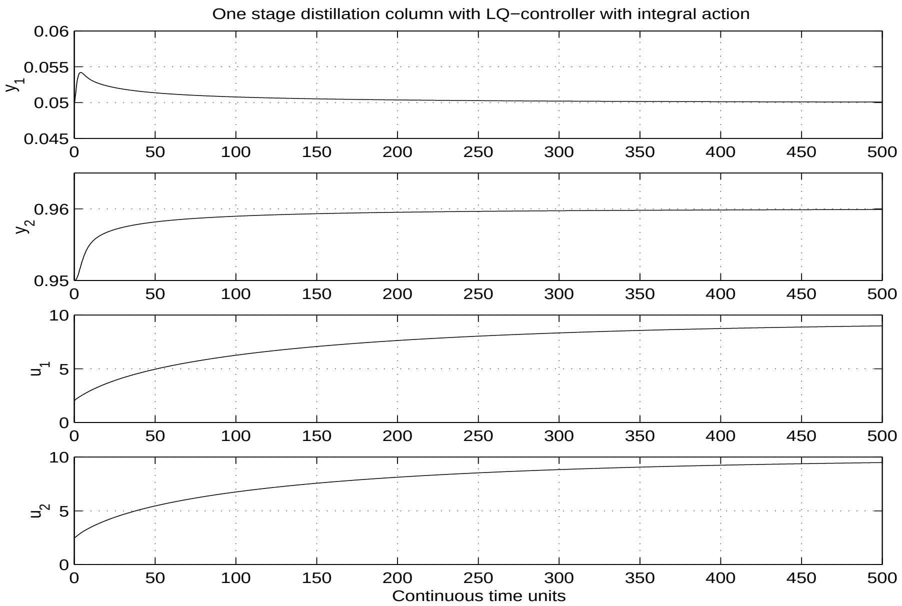

Figure 9.1: Simulation of the system in Example 9.2 with LQ optimal controller with integral action. The figure is generated with the MATLAB script **main_fcol3.m**.

The infinite time LQ optimal controller is of the form

$$u = g_1 x_1 + g_2 x_2. \tag{9.35}$$

A PD-controller can be written as

$$\begin{aligned} u &= K_p e + K_p T_d \dot{e} \\ &= K_p(-y) + K_p T_d(-\dot{y}) \\ &= K_p x_1 - K_p T_d \dot{x}_1, \end{aligned} \tag{9.36}$$

where we have used that $e = r - y = -y = -x_1$ when $r = 0$. This can be written as

$$u = -\frac{K_p}{1 + K_p T_d b_0} x_1 - \frac{K_p T_d}{1 + K_p T_d b_0} x_2. \tag{9.37}$$

The PD-controller parameters are then found by comparing (9.35) and (9.37) and solving for K_p and T_d . This gives

$$K_p = \frac{g_1}{1 + g_2 b_0}, \;\; T_d = \frac{g_2}{g_1}. \tag{9.38}$$

Hence, for 2nd order systems we have that the LQ optimal controller is equivalent with a PD-controller with optimal settings of the proportional gain, K_p, and the derivative time constant, T_d.

Example 9.4 (Equivalence between LQ and PD controllers)
Given a system on canonical observability form, say

$$\overbrace{\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix}}^{\dot{x}} = \overbrace{\begin{bmatrix} 0 & 1 \\ a_0 & a_1 \end{bmatrix}}^{A} \overbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}^{x} + \overbrace{\begin{bmatrix} 0 \\ 0.5 \end{bmatrix}}^{B} u, \tag{9.39}$$

$$y = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{x}, \tag{9.40}$$

where $a_0 = -2$ and $a_1 = -3$. The system have eigenvalues $\lambda_1 = -1$ and $\lambda_2 = -2$. Hence, the two time constants of the system is $T_1 = -\frac{1}{\lambda_1} = 1$ and $T_2 = -\frac{1}{\lambda_2} = 0.5$. The infinite time LQ optimal control problem with the following state and control input weighting matrices

$$Q = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \quad P = 1, \tag{9.41}$$

gives the following solution to the ARE, R, and the optimal feedback gain, G,

$$R = \begin{bmatrix} 1.5807 & 0.2462 \\ 0.2462 & 0.4085 \end{bmatrix}, \tag{9.42}$$

and

$$G = \begin{bmatrix} g_1 & g_2 \end{bmatrix} = \begin{bmatrix} -0.1231 & -0.2042 \end{bmatrix}. \tag{9.43}$$

This gives the LQ optimal control

$$u = Gx = g_1 x_1 + g_2 x_2, \tag{9.44}$$

where $g_1 = -0.1231$ and $g_2 = -0.2042$. Let us have a look at a standard PD-controller, i.e.,

$$u = K_p e + K_p T_d \dot{e}, \tag{9.45}$$

$$e = r - y. \tag{9.46}$$

where K_p is the proportional gain, T_d is the derivative time constant and r is the reference signal. From the state and output equations we have that

$$y = x_1, \tag{9.47}$$

$$\dot{e} = \dot{r} - \dot{y} = \dot{r} - \dot{x}_1 = \dot{r} - x_2. \tag{9.48}$$

Consider the case where $r = 0$, and substituting this into (9.45) gives.

$$u = -K_p x_1 - K_p T_d x_2, \tag{9.49}$$

Comparing the LQ controller (9.44)with the PD-controller (9.49) shows that they are equivalent if $g_1 = -K_p$ and $g_2 = -K_p T_d$, i.e.,

$$K_p = -g_1 = 0.1231, \tag{9.50}$$

$$T_d = \frac{g_1}{g_2} = 1.6590. \tag{9.51}$$

Hence, for this example the LQ optimal controller is equivalent with a PD-controller. However, note that this is not a general result, i.e., the result does not hold for nth order systems in general.

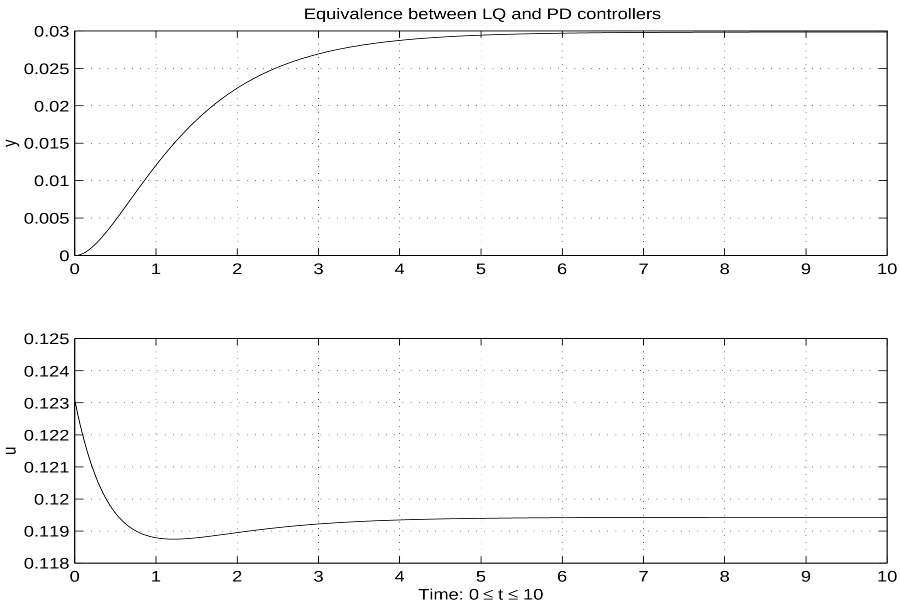

Figure 9.2: Simulation of the system in Example 9.4 with LQ optimal PD controller. The figure is generated with the MATLAB script **lq2pd_ex.m**, and with a unit step change at time zero in the reference signal r.

The system with the LQ optimal PD controller is implemented in the MATLAB script-file **lq2pd_ex.m**. *A simulation after a unit step change at time zero in the reference is illustrated in Figure 9.2. As we can see, there is a steady state error between the response in y and the reference signal, r. Hence, we have, as expected no integral action in the controller.*

Example 9.5 (Designing LQ optimal PID controller)
Given the system as in Example 9.4. Augmenting the state equation with the following model for the controller integrator

$$\dot{z} = r - y = r - Dx, \tag{9.52}$$

gives

$$\dot{\tilde{x}} = \overbrace{\begin{bmatrix} A & 0 \\ -D & 0 \end{bmatrix}}^{\tilde{A}} \tilde{x} + \overbrace{\begin{bmatrix} B \\ 0 \end{bmatrix}}^{\tilde{B}} u + \begin{bmatrix} 0 \\ 1 \end{bmatrix} r, \tag{9.53}$$

where

$$\tilde{x} = \begin{bmatrix} x \\ z \end{bmatrix}. \tag{9.54}$$

Choosing an LQ objective with infinite horizon, i.e.,

$$J = \frac{1}{2}\int_0^\infty (\tilde{x}^T \overbrace{\begin{bmatrix} Q_1 & 0 \\ 0 & Q_2 \end{bmatrix}}^{\tilde{Q}} \tilde{x} + u^T P u) dt, \tag{9.55}$$

with weighting matrices

$$Q = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \; Q_2 = 5, \; P = 1. \tag{9.56}$$

The LQ optimal controller is given by

$$u = Gx, \tag{9.57}$$
$$G = -P^{-1}\tilde{B}^T R, \tag{9.58}$$

where R is a solution to the ARE

$$\tilde{A}^T R + R\tilde{A} - R\tilde{B}P^{-1}\tilde{B}^T R + Q = 0. \tag{9.59}$$

Using e.g., the MATLAB function **are_schur.m** *gives the positive solution, R, to the ARE, and the optimal feedback gain matrix, G, as*

$$R = \begin{bmatrix} 20.7568 & 5.9406 & -15.7926 \\ 5.9406 & 2.1253 & -4.4721 \\ -15.7926 & -4.4721 & 15.5861 \end{bmatrix}, \; G = \begin{bmatrix} -2.9703 & -1.0627 & 2.2361 \end{bmatrix}. \tag{9.60}$$

This gives the LQ optimal controller

$$u = G\tilde{x} = g_1 x_1 + g_2 x_2 + g_3 z. \tag{9.61}$$

A PID controller can be written as

$$\begin{aligned} u &= K_p(r - y) + K_p T_d \dot{e} + \frac{K_p}{T_i} z \\ &= -K_p x_1 - K_p T_d x_2 + \frac{K_p}{T_i} z. \end{aligned} \tag{9.62}$$

Comparing with the LQ controller shows that they are equivalent if

$$K_p = -g_1 = 2.9703, \tag{9.63}$$
$$T_d = \frac{g_1}{g_2} = 0.3578, \tag{9.64}$$
$$T_i = -\frac{g_1}{g_3} = 1.3284. \tag{9.65}$$

The system with the LQ optimal PID controller is implemented in the MATLAB script-file **lq2pid_ex.m**. *A simulation after a unit step change at time zero in the reference is illustrated in Figure 9.3. As we can see, the response in y follows the reference with zero steady state error. Hence, we have integral action in the controller.*

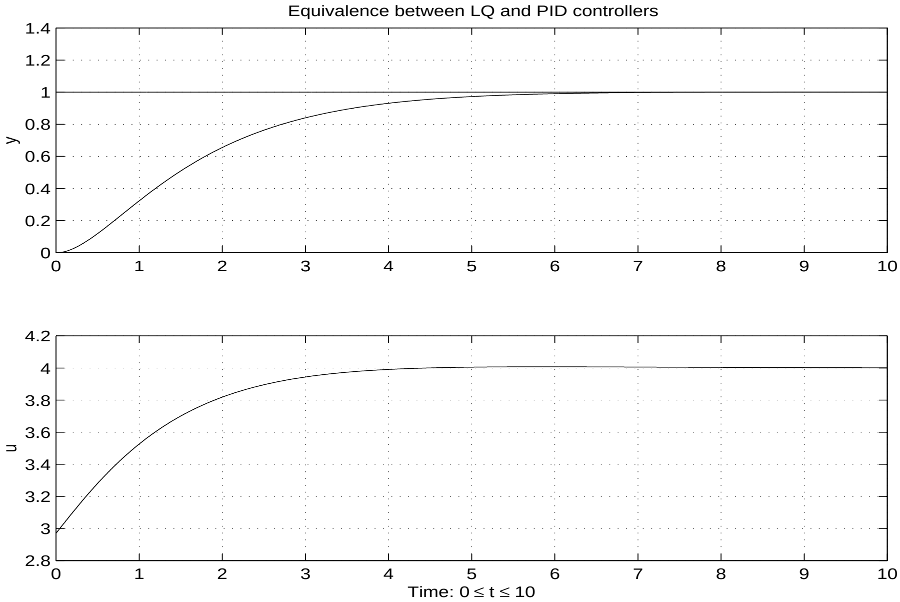

Figure 9.3: Simulation of the system in Example 9.5 with LQ optimal PID controller. The figure is generated with the MATLAB script **lq2pid_ex.m**, and with a unit step change at time zero for the reference signal r.

9.2 Matlab scripts for the examples

9.2.1 MATLAB script for Example 9.4

```
% lq2pd_ex.m
% Script for Example 3.4.
% This example shows the equivalence between an LQ optimal controller
% and a standard PD controller.
% Functions called: are_schur.
% Author: David Di Ruscio, 10.10.00.

%path(path,'s:\tex\fag\avreg\oving5\are_schur')

A=[0,1;-2 -3]; B=[0;0.5]; D=[1,0];    % The state space model for the process.
Q=[1,0;0,2]; P=1;                     % Weightings for the LQ objective.

R=are_schur(A,B,Q,P);                 % Solve the Riccati equation.
G=-inv(P)*B'*R

Kp=-G(1)                              % The equivalent PD parameters.
Td=G(2)/G(1)

t1=10; dt=0.01; t=0:dt:t1; N=length(t);

r=1;                                  % The reference signal.
x=[0;0];
```

```
for i=1:N
   y=D*x;
   u=-G(1)*(r-y)+G(2)*x(2);         % u=G*x written as a PD-controller.

   Y(i,1)=y;
   U(i,1)=u;

   dotx=A*x+B*u;
   x=x+dt*dotx;
end

subplot(211), plot(t,Y), ylabel('y'), grid
title('Equivalence between LQ and PD controllers')
subplot(212), plot(t,U), ylabel('u'), grid,
xlabel('Time: 0 \leq t \leq 10')

print -deps lq2pd_ex_fig                % Make figure in eps-format.
```

9.2.2 MATLAB script for Example 9.5

```
% lq2pid_ex.m
% Script for Example 3.5
% This example shows the equivalence between an LQ optimal controller
% and a standard PID controller.
% Functions called: are_schur.
% Author: David Di Ruscio, 10.10.00.

%path(path,'s:\tex\fag\avreg\oving5\are_schur')

A=[0,1;-2 -3]; B=[0;0.5]; D=[1,0];        % The state space model for the process.
Q=[1,0;0,2]; P=1; Q2=5;                   % Weighting matrix and parameters.
%Q=D'*10*D; P=1; Q2=5;                    % alternative weights.

At=[A,zeros(2,1);-D zeros(1)]; Bt=[B;0]; % Model for process with controller integrator
Qt=[Q zeros(2,1);zeros(1,2) Q2];          % The corresponding weighting matrix.

R=are_schur(At,Bt,Qt,P);                  % Solve the algebraic Riccati equation.
G=-inv(P)*Bt'*R                           % The LQ optimal feedback gain matrix.

Kp=-G(1)                                  % The equivalent parameters for the PID-contro
Ti=-G(1)/G(3)
Td=G(2)/G(1)

t1=10; dt=0.01;                           % Simulate the system.
t=0:dt:t1; N=length(t);

r=1;                                      % The reference signal.
x=[0;0]; z=0;                             % Initial values for the "states".
for i=1:N
   y=D*x;
   u=-G(1)*(r-y)+G(2)*x(2)+G(3)*z;        % u=G*x written as a PID-controller.

   Y(i,1)=y;
   U(i,1)=u;

   dotx=A*x+B*u;
   x=x+dt*dotx;                           % The process state.
   z=z+dt*(r-y);                          % the controller state (integrator).
end

subplot(211), plot(t,[r*ones(N,1) Y]), ylabel('y'), grid
title('Equivalence between LQ and PID controllers')
subplot(212), plot(t,U), ylabel('u'), grid
xlabel('Time: 0 \leq t \leq 10')

print -deps lq2pid_ex_fig                 % Make figure in eps-format.
```

Chapter 10

Examples on discrete time LQ optimal control

10.1 Examples: discrete time LQ-optimal control

Example 10.1 (LQ controller for scalar system)
Given a system described by the scalar system

$$x_{k+1} = ax_k + bu_k, \tag{10.1}$$
$$y_k = x_k, \tag{10.2}$$

with the following LQ objective function

$$J_i = \frac{1}{2}sy_N^2 + \frac{1}{2}\sum_{k=i}^{N-1}(qy_k^2 + pu_k^2). \tag{10.3}$$

The optimal control which minimizes the LQ objective is given by

$$g_k = -\frac{abr_{k+1}}{p + b^2 r_{k+1}}, \tag{10.4}$$

where r_{k+1} is given by the discrete time Riccati equation, i.e.,

$$r_k = q + a^2 r_{k+1} - \frac{a^2 b^2 r_{k+1}^2}{p + b^2 r_{k+1}}, \tag{10.5}$$
$$r_N = s. \tag{10.6}$$

Let $a = 0.9$, $b = 0.5$,$q = 2$, $p = 1$, $s = 2$, $i = 1$ and $N = 10$. A MATLAB script-file implementation of this example is illustrated in **ov7oppg3.m**.

10.2 Matlab scripts for the examples

10.2.1 MATLAB script for Example 10.1

```
% ov7oppg3.m
% Script for loesning av oppgave 3 paa oeving 7.
% David Di Ruscio.

a=0.9; b=0.5;                        % Modellparametre.
q=2; p=1; s=2;                       % Vektparametre.
x0=10;                               % Initialverdi for tilstanden.
N=10;                                % Slutt-tid (diskret tidspunkt).

R=zeros(N,1);                        % Setter av plass for lsningene av Riccati-lignige
G=zeros(N-1,1);                      % Setter av plass for tilbakekoplingskoeffisientene
r=s;                                 % Grensebetingelse, R_N=S_N.
R(N)=s;
for k=N-1:-1:1                       % Itererer fra slutt-tiden til initial-tidspunktet.
   k
   g=-b*a*r/(p+b^2*r);               % Optimal tilbakekoplings-konstant, g(k)=f(r(k+1)).
   r=a^2*r-a^2*b^2*r^2/(p+b^2*r)+q; % Den skalare Riccati-ligning, r(k)=f(r(k+1)).
   R(k)=r;
   G(k)=g;
end

Y=zeros(N,1); U=zeros(N-1,1);        % Simulerer systemet med optimal LQ-regulator.
x=x0;                                % Initialverdi for tilstanden.
for k=1:N-1
   y=x;
   Y(k)=x;
   u=G(k)*x;
   U(k)=u;
   x=a*x+b*u;
end
Y(N)=x;

Ys=zeros(N,1); Us=zeros(N-1,1);      % Simulerer systemet med suboptimal LQ-regulator.
x=x0;                                % Initialverdi for tilstandsvektoren.
for k=1:N-1
   y=x;
   Ys(k)=x;
   u=G(1)*x;
   Us(k)=u;
   x=a*x+b*u;
end
Ys(N)=x;

figure(1)
```

```
subplot(211), plot(1:length(Y),Y,'bo-'), ylabel('y_k')
subplot(212), plot(1:length(U),U,'bo-'), ylabel('u_k')

figure(2)
subplot(211), plot(1:length(R),R,'bo-'), ylabel('r_k')
subplot(212), plot(1:length(G),G,'bo-'), ylabel('g_k')
xlabel('Discrete time: 1 \leq k \leq 10')
```

Part III

ESTIMATION AND CONTROL

Chapter 11

Control and Estimation

11.1 Continuous estimator and regulator duality

It can be shown that the solution to the Linear Quadratic optimal control problem is dual to the optimal minimum variance estimator problem, Kalman filter. This means that if we know the solution to the LQ optimal control problem, then we can directly write down the solution to the optimal estimator problem by using the duality principle. However, note that the LQ optimal control problem is a topic of a course in Advanced control theory.

The duality principle can be formulated in the following table

$$
\begin{array}{ccc}
\text{Regulator} & & \text{Estimator} \\
A & \rightarrow & A^T \\
B & \rightarrow & D^T \\
Q & \rightarrow & V \\
P & \rightarrow & W \\
G & \rightarrow & -K^T \\
A + BG & \rightarrow & (A^T - D^T K^T)^T \\
R & \rightarrow & X \\
-t & \rightarrow & t \\
\dot{R} & \rightarrow & -\dot{X}
\end{array}
\tag{11.1}
$$

As we know from the solution of the LQ optimal control problem the Riccati equation is solved backward in time from the final time instant, i.e. recursively from the final value, $R(t_1) = S$. The solution to the dual minimum variance estimator problem is also containing a Riccati equation. The Riccati equation in the dual estimator problem is however solved forward in time with initial values given at the start time. This is the reason why we have specified $-t$ in the table for the LQ control problem and t in connection with the dual estimator problem.

11.2 Minimum variance estimation in linear continuous systems

Given a linear dynamic system described by

$$\dot{x} = Ax + Bu + v, \tag{11.2}$$
$$y = Dx + Eu + w, \tag{11.3}$$

where v is uncorrelated white process noise with zero mean and covariance matrix V and w is uncorrelated white measurements noise with zero mean and covariance matrix W, i.e. such that

$$V = \mathrm{E}(vv^T), \tag{11.4}$$
$$W = \mathrm{E}(ww^T). \tag{11.5}$$

Furthermore, in this section we assume the process noise v to be uncorrelated/independent of the measurements noise w, i.e. $\mathrm{E}(vw^T) = 0$. We assume that A, B, D and E are known model matrices. Furthermore we assume that the covariance matrices V and W are known or specified and that the measurements vector y is measured and given. We also assume that the matrix pair A, D is observable. Since the state vector x is not measured it can be estimated in a so called state estimator or state observer.

The principle of duality in connection with the solution of the Linear Quadratic (LQ) optimal control problem can be used to find the solution to the optimal minimum variance estimation problem.

Note that we have from the duality principle that $\dot{R} \rightarrow \frac{dX}{d(-t)} = -\dot{X}$. using the duality principle we have that

$$\dot{X} = AX + XA^T - XD^TW^{-1}DX + V, \;\; X(t_0) \text{ given}, \tag{11.6}$$

which is a matrix Riccati equation which defines X. The Kalman filter gain matrix is then given by

$$K^T = W^{-1}DX. \tag{11.7}$$

Let us define the error between the actual state, x, and the estimated state, $\hat{x}$, as follows

$$\Delta x = x - \hat{x}. \tag{11.8}$$

It can be shown that the solution to the riccati equation, X, is the covariance matrix of the error between x and the estimate $\hat{x}$, i.e.

$$X = \mathrm{E}[(x - \hat{x})(x - \hat{x})^T] = \mathrm{E}[\Delta x \Delta x^T]. \tag{11.9}$$

The state estimator is then given by

$$\dot{\hat{x}} = A\hat{x} + Bu + K(y - \hat{y}), \tag{11.10}$$
$$\hat{y} = D\hat{x} + Eu. \tag{11.11}$$

$\hat{x}$ is the minimum variance estimate of the state vector x in the sense that X is minimized. Note also that $\hat{y}$ is the optimal prediction of the measurements vector y, given all old outputs y and given all old input vectors u as vell as the present input at the present time t.

The reason for that $\hat{y}$ is dependent of the input u at present time t is the direct feed through term matrix E. However E is in principle always zero for continuous systems, but a nonzero E may be the results of some model reduction procedures. Note also that a non zero E often is the case in discrete time systems due to sampling.

Equations (11.10) and (11.11) gives the following equation for the state estimate

$$\dot{\hat{x}} = (A - KD)\hat{x} + (B - KE)u + Ky, \tag{11.12}$$

where the initial state estimate $\hat{x}(t_0)$ is given.

Note that the eigenvalues of the matrix $A - KD$ defines the stability properties of the estimator. It make sense that K is so that $A - KD$ is stable, i.e., all eigenvalues in the left half of the complex plane. the reason for this is that $\hat{x}$ is given from a differential equation driven by known inputs u and known outputs y. Note also that when $A - kD$ is stable then the effect of wrong initial values $\hat{x}(t_0)$ will die out when $t \to \infty$.

Let us study the properties of the estimator by studying the excepted value of the error in the state estimate Δx. From the definition (11.8) we have that

$$\dot{\Delta x} = \dot{x} - \dot{\hat{x}}. \tag{11.13}$$

Using (11.2) and (11.10) gives

$$\dot{\Delta x} = Ax + Bu + v - [A\hat{x} + Bu + K(y - \hat{y})]. \tag{11.14}$$

using (11.3) and (11.11) gives

$$\dot{\Delta x} = Ax + Bu + v - [A\hat{x} + Bu + K(Dx + Eu + w - D\hat{x} - Eu)], \tag{11.15}$$

which gives

$$\dot{\Delta x} = (A - KD)\Delta x + v - Kw. \tag{11.16}$$

The excepted value of the estimated error, Δx, is then given by

$$\mathrm{E}\{\dot{\Delta x}\} = (A - KD)\mathrm{E}\{\Delta x\}. \tag{11.17}$$

The stability properties of the estimator can be analyzed by studying the estimation error when $t \to \infty$.

It can be shown that the minimum variance estimator is stable. This can be argued from the fact that the LQ optimal controller is stable (by properly choice of some weighting matrices) and that the optimal minimum variance estimator is dual to the LQ controller. Hence, a similar stability theorem exists for the optimal minimum variance estimator.

In the following a different argumentation for stability will be given. Assume that v and w is uncorrelated white noise stationary processes. Then the covariance matrices will be constant and positive definite, i.e., $V > 0$ and $W > 0$. Letting $t \to \infty$ then we have that X is a solution to the stationary algebraic matrix Riccati equation

$$AX + XA^T - XD^TW^{-1}DX + V = 0. \tag{11.18}$$

This can be written as a Lyapunov matrix equation, i.e.,

$$(A - KD)X + X(A - KD)^T = -(V + KWK^T). \tag{11.19}$$

From the discussion above it is clear that $X > 0$ and $V + KWK^T > 0$. From Lyapunovs stability theory we then know that $A - KD$ is a stable matrix, i.e. all eigenvalues of $A - KD$ lies in the left half of the complex plane.

It is clear that when $A-KD$ is a stable matrix then the excepted value is $\mathrm{E}\{\dot{\Delta x}\} = 0$. From (11.17) we then have that $0 = (A-KD)\mathrm{E}\{\Delta x\}$. This implies that $\mathrm{E}\{\Delta x\} = 0$.

Another alternative is to analyze the error from the solution of (11.17). We have

$$\lim_{t\to\infty} \mathrm{E}\{\Delta x\} = \lim_{t\to\infty} [e^{(A-KD)(t-t_0)}]\mathrm{E}\{\Delta x(t_0)\} = 0, \tag{11.20}$$

which is valid even if $\mathrm{E}\{\Delta x(t_0)\} \neq 0$.

11.3 Separation Principle: Continuous time

Theorem 11.3.1 (Separation Principle)
Given a linear continuous time combined deterministic and stochastic system

$$\dot{x} = Ax + Bu + Cv, \tag{11.21}$$
$$y = Dx + w, \tag{11.22}$$

where v and w is uncorrelated zero mean white noise processes with covariance matrices V og W, respectively.

The system should be controlled such that the following performance index is minimized

$$J = \frac{1}{2}\mathrm{E}\{x^T(t_1)Sx(t_1) + \int_{t_0}^{t_1} [x^TQx + u^TPu]dt\}, \tag{11.23}$$

with respect to the control vector $u(t)$ in time interval, $t_0 \leq t < t_1$.

The solution to this stochastic optimal control problem is given by

$$u = G(t)\hat{x}. \tag{11.24}$$

G is the feedback gain matrix found by solving the corresponding deterministic LQ optimal control problem where x is known, i.e., with $v = 0$ and $w = 0$ in (11.21) and (11.22) and the same LQ objective as in (11.23). It is no need for the expectation operator $\mathrm{E}(\{\cdot\}$ in the deterministic case. This means that G is given by

$$G(t) = -P^{-1}B^TR \tag{11.25}$$

where R is the unique positive definite solution to the-equation

$$-\dot{R} = A^T R + RA - RBP^{-1}B^T R + Q, \;\; R(t_1) = S. \tag{11.26}$$

$\hat{x}$ is optimal minimum variance estimate of the state vector x. $\hat{x}$ is given by the Kalman-filter for the system, given by

$$\dot{\hat{x}} = A\hat{x} + Bu + K(y - D\hat{x}), \tag{11.27}$$

with given initial state, $\hat{x}(t_0)$, and where the Kalman filter gain matrix, K, is given by

$$K(t) = XD^T W^{-1}, \tag{11.28}$$

and where X is the maximum positive definite solution to the Riccati equation

$$\dot{X} = AX + XA^T - XD^T W^{-1} DX + CVC^T, \;\; X(t_0) = \text{given}. \tag{11.29}$$

$\triangle$

Often an infinite time horizon is used, i.e., $t_1 \to \infty$. This leads to the stationary Riccati equation, i.e., putting ($\dot{R} = 0$) and the stationary Riccati equation for X, i.e., with $\dot{X} = 0$ i (11.29). In this case the gain matrices G and K are constant time invariant matrices. Note that a stationary Riccati equation are denoted an Algebraic Riccati Equation (ARE).

11.4 Continuous LQG controller

An Linear Quadratic Gausian (LQG) controller for MIMO systems where an Linear Quadratic (LQ) optimal feedback matrix G is used in a feedback from the minimum variance optimal (Kalman filter) estimate, $\hat{x}$, of the process/system state x. The controller is basically of the form $u = G\hat{x}$. The LQG controller may be useful in problems where the state vector x is not measured or available.

A short description of the LQG controller is as follows. Given a system model

$$\dot{x} = Ax + Bu, \tag{11.30}$$
$$y = Dx, \tag{11.31}$$

and the state observer

$$\dot{\hat{x}} = A\hat{x} + Bu + K(y - \hat{y}), \tag{11.32}$$
$$\hat{y} = D\hat{x}, \tag{11.33}$$

and the controller

$$u = G\hat{x}. \tag{11.34}$$

An analysis of the total closed loop system with LQG controller is as follows. Note that the analysis is valid for arbitrarily gain matrices G and K.

The above Equations (11.30)-(11.34) gives an autonomous system

$$\begin{bmatrix} \dot{x} \\ \dot{\hat{x}} \end{bmatrix} = \begin{bmatrix} A & BG \\ KD & A + BG - KD \end{bmatrix} \begin{bmatrix} x \\ \hat{x} \end{bmatrix}. \tag{11.35}$$

The stability of the total system is given by the eigenvalues of the system matrix. For simplicity of stability analysis we study the transformed system, i.e.,

$$\begin{bmatrix} x \\ x - \hat{x} \end{bmatrix} = \begin{bmatrix} x \\ \Delta x \end{bmatrix} = \begin{bmatrix} I & 0 \\ I & -I \end{bmatrix} \begin{bmatrix} x \\ \hat{x} \end{bmatrix}. \tag{11.36}$$

this gives the autonomous system

$$\begin{bmatrix} \dot{x} \\ \dot{\Delta x} \end{bmatrix} = \overbrace{\begin{bmatrix} A + BG & -BG \\ 0 & A - KD \end{bmatrix}}^{\bar{A}_{tc}} \begin{bmatrix} x \\ \Delta x \end{bmatrix}. \tag{11.37}$$

because

$$\begin{bmatrix} I & 0 \\ I & -I \end{bmatrix}^{-1} = \begin{bmatrix} I & 0 \\ I & -I \end{bmatrix}. \tag{11.38}$$

As we see, the stability of the entire LQG controlled system is given by n eigenvalues from the "feedback" matrix $A + BG$ and n eigenvalues from the "estimator" matrix $A - KD$. The LQG system matrix $\bar{A}_{tc}$ have $2n$ eigenvalues.

As a rule of thumb the estimator gain matrix K is "tuned" such that the eigenvalues of the matrix $A - KD$ lies to the left of the eigenvalues of $A + BG$ in the left half part of the complex plane. Often it is stated that the time constants of the estimator $A - KD$ should be approximately ten times faster than the time constants of the matrix $A + BG$.

If we have modeling errors then the LQG controller should be analyzed for robustnes. It may be shown that the LQG controlled system may be unstable due to modeling errors, and an LQG design should always be analyzed for robustness (stability) due to perturbations (errors) in the model.

One should that an LQG controller is close to an MPC controller and the same robustness/stability analysis due to modeling errors should be performed for any model based controller in which an estimate $\hat{x}$ is used for feedback instead of the actual state x.

11.5 Discrete time LQG controller

11.5.1 Analysis of discrete time LQG controller

We will in this section discuss the discrete time LQG controller. We assume that the process is described by

$$x_{k+1} = Ax_k + B_p u_k, \tag{11.39}$$
$$y_k = Dx_k. \tag{11.40}$$

The controller is of the form

$$u_k = G\hat{x}_k. \tag{11.41}$$

where $\hat{x}_k$ is given by the state observer

$$\bar{y}_k = D\bar{x}_k \tag{11.42}$$
$$\hat{x}_k = \bar{x}_k + K(y_k - \bar{y}_k), \tag{11.43}$$
$$\bar{x}_{k+1} = A\hat{x}_k + Bu_k. \tag{11.44}$$

where $\bar{x}_0$ is given. Here $\bar{x}_k$ is defined as the a-priori estimate of x_k. Furthermore we define $\hat{x}_k$ as the a-posteriori estimate of x_k. We assume that the feedback matrix G is computed based on the model matrices A, B. The observer gain matrix K is computed based on the model matrices A, D.

We see that we have a perfect model is $B = B_p$. If $B \neq B_p$ then we have modeling errors. Let us in the following study the entire closed loop system. Putting (11.41) into (11.39) and (11.44) and we obtain

$$x_{k+1} = Ax_k + B_pG\hat{x}_k, \tag{11.45}$$
$$\bar{x}_{k+1} = (A + BG)\hat{x}_k. \tag{11.46}$$

We may now eliminate $\hat{x}_k$ from (11.45) and (11.46) by using (11.43).

$$x_{k+1} = (A + B_pGKD)x_k + B_pG(I - KD)\bar{x}_k, \tag{11.47}$$
$$\bar{x}_{k+1} = (A + BG)KDx_k + (A + BG)(I - KD)\bar{x}_k. \tag{11.48}$$

This means that we have an autonomous system

$$\begin{bmatrix} x_{k+1} \\ \bar{x}_{k+1} \end{bmatrix} = \overbrace{\begin{bmatrix} A + B_pGKD & B_pG(I - KD) \\ (A + BG)KD & (A + BG)(I - KD) \end{bmatrix}}^{A_{td}} \begin{bmatrix} x_k \\ \bar{x}_k \end{bmatrix}. \tag{11.49}$$

The entire system is stable if the $2n$ eigenvalues of the matrix A_{td} is located inside the unit circle in the complex plane. Let us use the transformation (11.36). This gives

$$\begin{bmatrix} x_{k+1} \\ x_{k+1} - \bar{x}_{k+1} \end{bmatrix} = \overbrace{\begin{bmatrix} A + B_pG & -B_pG(I - KD) \\ (B_p - B)G & A - AKD - (B_p - B)G(I - KD) \end{bmatrix}}^{\bar{A}_{td}} \begin{bmatrix} x_k \\ x_k - \bar{x}_k \end{bmatrix} \tag{11.50}$$

In case of a perfect model, i.e., $B = B_p$, we se that the eigenvalues of the total system is given by the n eigenvalues of the matrix $A + BG$ and the n egenvalues of the observer system matrix $A - AKD$.

This also means that in case of modeling errors we have to check the eigenvalues/poles of the system matrix for the entire system, i.e., $\bar{A}_{td}$ for different cases of model errors B_p.

Note also that a rule of thumb is that the eigenvalues of the observer matrix $A - AKD$ should be ten times faster than the eigenvalues of the controller feedback matrix $A + BG$.

11.6 The discrete Kalman filter

11.6.1 Innovation formulation of the Kalman filter

Given a process

$$x_{k+1} = Ax_k + Bu_k + v_k, \tag{11.51}$$
$$y_k = Dx_k + w_k, \tag{11.52}$$

where v_k is white process noise and w_k is white measurements noise with known covariance matrices.

First, let us present the apriori-aposteriori formulation of the discrete time optimal minimum variance Kalman filter as follows

$$\bar{y}_k = D\bar{x}_k \tag{11.53}$$
$$\hat{x}_k = \bar{x}_k + K(y_k - \bar{y}_k), \tag{11.54}$$
$$\bar{x}_{k+1} = A\hat{x}_k + Bu_k. \tag{11.55}$$

where $\bar{x}_0$ is a given initial value for the apriori or predicted state estimate. Here, $\bar{x}_k$ is defined as the apriori or predicted state estimate of the state vector x_k. Furthermore, $\hat{x}_k$ is defined as the aposteriori state estimate of x_k. The apriori-aposteriori Kalman filter is further discussed in Section 11.6.3.

Note that $\hat{x}_k$ can be eliminated from the estimator equation (11.55), i.e. an equivalent estimator for the predicted state $\bar{x}_k$ is given by

$$\bar{y}_k = D\bar{x}_k, \tag{11.56}$$
$$\bar{x}_{k+1} = A\bar{x}_k + Bu_k + \tilde{K}(y_k - \bar{y}_k). \tag{11.57}$$
$$= (A - \tilde{K}D)\bar{x}_k + Bu_k + \tilde{K}y_k, \tag{11.58}$$

where

$$\tilde{K} = AK. \tag{11.59}$$

It is the apriori estimate, $\bar{x}_k$ which is the essential state in the estimator. $\bar{x}_k$ is also referred to as the predicted state.

The dynamics of the estimator is in this case described by the eigenvalues of the matrix $A - \tilde{K}D = A - AKD$. the estimator given by (11.56)-(11.57) above gives the optimal one step ahead prediction $\bar{y}_k$ of the output y_k. This formulation is used if we only want to compute the prediction of the output y_k. As a rule of thumb we may say that $\tilde{K} = AK$ is the Kalman filter gain for the prediction of y_k and for computing the predicted state $\bar{x}_k$.

Note also that if we are using $y_k = \bar{y}_k + \varepsilon_k$ where the predicted output is given by $\bar{y}_k = D\bar{x}_k$ then we obtain the innovations formulation of the Kalman filter, i.e.,

$$\bar{x}_{k+1} = A\bar{x}_k + Bu_k + \tilde{K}\varepsilon_k, \tag{11.60}$$
$$y_k = D\bar{x}_k + \varepsilon_k, \tag{11.61}$$

where $\varepsilon_k = y_k - \bar{y}_k$ is the innovations process.

Notice that the optimal Kalman filter gain $\tilde{K}$ is such that the innovations process ε_k is white noise.

This means that $\tilde{K} = AK$ is the kalman filter gain in the innovations formulation (11.60)-(11.61) and K is the Kalman filter gain in the apriori-aposteriori formulation (11.42)-(11.44) of the Kalman filter.

Note that the above equations easily is extended to be valid for a proper system in which $y_k = D\bar{x}_k + Eu_k + \varepsilon_k$.

11.6.2 Development of the Kalman filter on innovations form

Given a process

$$x_{k+1} = Ax_k + v_k, \tag{11.62}$$

$$y_k = Dx_k + w_k, \tag{11.63}$$

where v_k is white process noise and w_k is white measurements noise with covariance matrices given by

$$\mathrm{E}(\begin{bmatrix} v_k \\ w_k \end{bmatrix} \begin{bmatrix} v_k \\ w_k \end{bmatrix}^T) = \begin{bmatrix} V & R_{12} \\ R_{12}^T & W \end{bmatrix} \tag{11.64}$$

The Kalman filter on innovations form is then given by

$$\bar{x}_{k+1} = A\bar{x}_k + \tilde{K}\varepsilon_k, \tag{11.65}$$

$$y_k = D\bar{x}_k + \varepsilon_k. \tag{11.66}$$

Note that the Kalman filter gain $\tilde{K}$ in the innovations formulation is related to the Kalman filter gain K in the apriori-aposteriori formulation as $\tilde{K} = AK$.

When analyzing the Kalman filter the estimating error $\Delta x_k = x_k - \bar{x}_k$ is of great importance. The equations for the estimating errors are obtained from the above equations. i.e. from the process model and the Kalman filter above, i.e.,

$$\Delta x_{k+1} = A\Delta x_k + v_k - \tilde{K}\varepsilon_k, \tag{11.67}$$

$$\varepsilon_k = D\Delta x_k + w_k, \tag{11.68}$$

$$\Delta x_k = x_k - \bar{x}_k. \tag{11.69}$$

The equations for the estimating error are to be used in the following discussions.

Equation for computing $\tilde{K}$ in the predictor

The development which is given here is based on the fact that the innovations process ε_k is white noise when the optimal Kalman filter gain $\tilde{K}$ is used in the filter. Since ε_k is white it is independent and uncorrelated with the estimation error Δx_{k+1}. Hence, by demanding

$$\mathrm{E}(\Delta x_{k+1}\varepsilon_k^T) = 0, \tag{11.70}$$

then we can derive an expression for $\tilde{K}$. We have that

$$\begin{aligned}\Delta x_{k+1}\varepsilon_k^T &= (A\Delta x_k + v_k - \tilde{K}\varepsilon_k)\varepsilon_k^T \\ &= A\Delta x_k\varepsilon_k^T + v_k\varepsilon_k^T - \tilde{K}\varepsilon_k\varepsilon_k^T \\ &= A\Delta x_k(\Delta x_k^T D^T + w_k^T) + v_k(\Delta x_k^T D^T + w_k^T) - K\varepsilon_k\varepsilon_k^T. \quad (11.71)\end{aligned}$$

Using this in (11.70) gives

$$\mathrm{E}(A\Delta x_k\Delta x_k^T D^T + v_k w_k^T - \tilde{K}\varepsilon_k\varepsilon_k^T) = 0, \quad (11.72)$$

where we have used that $\mathrm{E}(\Delta x_k v_k^T) = 0$ and $\mathrm{E}(\Delta x_k w_k^T) = 0$. We have then obtained an equation

$$AXD^T + R_{12} - \tilde{K}\Delta = 0, \quad (11.73)$$

where

$$\Delta = \mathrm{E}(\varepsilon_k\varepsilon_k^T) = DXD^T + W. \quad (11.74)$$

This gives the following expression for the Kalman filter gain

$$\tilde{K} = (AXD^T + R_{12})(DXD^T + W)^{-1}. \quad (11.75)$$

This is the equation for th Kalman filter gain in the innovations formulation of the Kalman filter. We now have to find an expression for the covariance matrix of the estimation error, $X = \mathrm{E}(\Delta x_k\Delta x_k^T)$. It can be shown that X is given as the solution of a matrix Riccati equation.

Equation for computing $X = \mathbf{E}(\Delta x_k\Delta x_k^T)$

The derivation of the riccati equation for computing the covariance matrix X is based that we under stationary conditions have that

$$\mathrm{E}(\Delta x_{k+1}\Delta x_{k+1}^T) = \mathrm{E}(\Delta x_k\Delta x_k^T) = X. \quad (11.76)$$

From equations (11.67) and (11.68) we have that

$$\Delta x_{k+1} = A\Delta x_k + v_k - \tilde{K}\overbrace{(D\Delta x_k + w_k)}^{\varepsilon_k}, \quad (11.77)$$

which gives

$$\Delta x_{k+1} = (A - \tilde{K}D)\Delta x_k + v_k - \tilde{K}w_k. \quad (11.78)$$

we have that the estimation error Δx_k is uncorrelated with the white noise processes v_k and w_k. We then have that

$$\begin{aligned}\Delta x_{k+1}\Delta x_{k+1}^T &= [(A - \tilde{K}D)\Delta x_k + v_k - \tilde{K}w_k][(A - \tilde{K}D)\Delta x_k + v_k - \tilde{K}w_k]^T \\ &= (A - \tilde{K}D)\Delta x_k\Delta x_k^T(A - \tilde{K}D)^T + (v_k - \tilde{K}w_k)(v_k - \tilde{K}w_k)^T \\ &= (A - \tilde{K}D)\Delta x_k\Delta x_k^T(A - \tilde{K}D)^T + v_k v_k^T - v_k w_k^T\tilde{K}^T \\ &- \tilde{K}(v_k w_k^T)^T + \tilde{K}w_k w_k^T\tilde{K}^T. \quad (11.79)\end{aligned}$$

Using the mean operator E(·) on both sides of the equal sign gives

$$X = (A - \tilde{K}D)X(A - \tilde{K}D)^T + V - R_{12}\tilde{K}^T - \tilde{K}R_{12}^T + \tilde{K}W\tilde{K}^T, \quad (11.80)$$

which also can be written as follows

$$X = (A - \tilde{K}D)X(A - \tilde{K}D)^T + \begin{bmatrix} I & \tilde{K} \end{bmatrix} \begin{bmatrix} V & R_{12} \\ R_{12}^T & W \end{bmatrix} \begin{bmatrix} I & \tilde{K} \end{bmatrix}^T. \quad (11.81)$$

Note that (11.80) and (11.81) is a discrete matrix Lyapunov equation in X when $\tilde{K}$ is given. A Lyapunov equation is a linear equation. The Lyapunov equation can e.g. simply be solved by using the MATLAB control system toolbox function **dlyap**. By substituting the expression for the Kalman filter gain $\tilde{K}$ given by (11.75) into (11.81) gives the discrete Riccati equation for computing the covariance matrix X, i.e.,

$$\begin{aligned} X &= AXA^T + V - \tilde{K}(AXD^T + R_{12})^T \\ &= AXA^T + V - (AXD^T + R_{12})(DXD^T + W)^{-1}(AXD^T + R_{12})^T. \end{aligned} \quad (11.82)$$

The stationar Riccati equation can simply be solved for X by iterating (11.82) until convergence. Another elegant method is to iterate both (11.75) and (11.80) until convergence and computing both $\tilde{K}$ and X at the same time. this is illustrated and implemented in the MATLAB function **dlqe2.m**.

```
function [K,X,itnum]=dlqe2(A,C,D,V,W,R12);
% DLQE2
% [K,X]=dlqe2(A,C,D,V,W,R12);
% This function computes the Kalman gain K in the Kalman filter on
% innovations form, and the covariance matrix X of the estimation
% error, i.e. the error between the state and the predicted state.

X=C*V*C';                          % Initial covariance matrix.
K=(A*X*D'+R12)*pinv(D*X*D'+W);     % The corresponding Kalman gain.
it=100;                            % Maximum number of iterations.
Tol=1e-8;                          % Tolerance for norm(X(i)-X(i-1)).

Xold=X*0;                          % Iterate for the solution X of
for i=1:it;                        % the discrete Riccati equation.
   K=(A*X*D'+R12)*pinv(D*X*D'+W);
   AKD=A-K*D;
   X=AKD*X*AKD'+V-R12*K'-K*R12'+K*W*K';
   if norm(X-Xold) <= Tol
      itnum=i;
      break
   end
   Xold=X;
end
K=(A*X*D'+R12)*pinv(D*X*D'+W);
```

11.6.3 Derivation of the Kalman filter on apriori-aposteriori form

Given a process

$$x_{k+1} = Ax_k + v_k, \tag{11.83}$$
$$y_k = Dx_k + w_k, \tag{11.84}$$

where v_k is white process noise and w_k is white measurements noise with covariance matrices given by

$$\mathrm{E}(\begin{bmatrix} v_k \\ w_k \end{bmatrix} \begin{bmatrix} v_k \\ w_k \end{bmatrix}^T) = \begin{bmatrix} V & R_{12} \\ R_{12}^T & W \end{bmatrix}. \tag{11.85}$$

We here note that the process noise v_k may be correlated with the measurements noise w_k, i.e. $\mathrm{E}(v_k w_k^T) = R_{12}$.

The kalman filter on apriori-aposteriori form is basically used when we are out for the optimal state estimate of x_k. The filter is of the form

$$\bar{y}_k = D\bar{x}_k \tag{11.86}$$
$$\hat{x}_k = \bar{x}_k + K(y_k - \bar{y}_k), \tag{11.87}$$
$$\bar{x}_{k+1} = A\hat{x}_k + R_{12}\Delta^{-1}(y_k - \bar{y}_)), \tag{11.88}$$

where the initial predicted state $\bar{x}_0$ is given or specified. Here $\bar{x}_k$ is defined as the apriori state estimate of x_k. the estimate $\bar{x}_k$ is also often referred to as the predicted state. Furthermore we define $\hat{x}_k$ as the aposteriori state estimate of x_k. Apriori means known in advance, and aposteriori means the new information which is obtained by the updating in (11.87), i.e., by using the apriori information and the new information in the measurement y_k. The reason for that the state estimate is divided into two parts $\bar{x}_k$ and $\hat{x}_k$ is mainly because the system is discrete time, e.g. because of sampling.

The kalman filter gain K in the filter given by (11.86)-(11.88) above is given by

$$K_k = \bar{X}_k D^T (D\bar{X}_k D^T + W)^{-1}, \tag{11.89}$$
$$\hat{X}_k = (I - K_k D)\bar{X}_k (I - K_k D)^T + K_k W K_k^T, \tag{11.90}$$
$$\bar{X}_{k+1} = A\hat{X}_k A^T + V + Z_k, \tag{11.91}$$

where

$$Z_k = -R_{12}\Delta^{-1}R_{12}^T - AK_k R_{12}^T - R_{12}K_k^T A^T. \tag{11.92}$$

Note that (11.91) contain an extra term given by Z_k when the process and measurements noise is correlated, this term is not present when $R_{12} = 0$, which usually is the case.

In order to start the filter process we need an initial value for the covariance matrix $\bar{X}_0$, i.e. when we look at the filter at time $k = 0$. Note that the covariance matrices are defined as follows

$$\bar{X}_k = \mathrm{E}((x_k - \bar{x}_k)(x_k - \bar{x}_k)^T), \tag{11.93}$$
$$\hat{X}_k = \mathrm{E}((x_k - \hat{x}_k)(x_k - \hat{x}_k)^T). \tag{11.94}$$

Note that when the system is time invariant, i.e. when the system matrices A and D and the noise covariance matrices V, W og R_{12} are constant matrices, then the filter will be stationary and we will have that $\bar{X}_{k+1} = \bar{X}_k = \bar{X}$ and $K_k = K$ are constant matrices. Note also that (11.90) can be expressed as the following alternative

$$\hat{X}_k = \bar{X}_k - K_k D \bar{X}_k. \tag{11.95}$$

However, Equation (11.90) is to be preferred of numerical reasons due to the fact that all terms in (11.90) are symmetric and positive semidefinite. Hence, it is of higher probability that the final computed results is symmetric and positive semidefinite by using (11.90). The final computed covariance matrix $\hat{X}$ should be symmetric and positive semidefinite, i.e. symmetric and $\hat{X} \geq 0$

Equation for computing K_k in the filter

The derivation of the Kalman filter gain matrix presented in this section is based on the fact that when K_k is the optimal minimum variance filter gain, then the innovations process, ε_k, is white noise and uncorrelated with the state deviation variables $\Delta\bar{x}_{k+1} = x_{k+1} - \bar{x}_{k+1}$ as well as $\Delta\hat{x}_k = x_k - \hat{x}_k$, i.e.,

$$\begin{aligned} \mathrm{E}(\Delta\bar{x}_{k+1}\varepsilon_k^T) &= \mathrm{E}((x_{k+1} - \bar{x}_{k+1})\varepsilon_k^T) \\ &= \mathrm{E}((Ax_k + v_k - A\hat{x}_k)\varepsilon_k^T) = A\mathrm{E}(\Delta\hat{x}_k\varepsilon_k^T) = 0 \ , \end{aligned} \tag{11.96}$$

since $\mathrm{E}(v_k\varepsilon_k^T) = 0$

In this section we will derive an expression for the stationary Kalman filter gain, K, from the equation

$$\mathrm{E}(\Delta\hat{x}_k\varepsilon_k^T) = 0. \tag{11.97}$$

We take the updating given by (11.87) as the starting point and write

$$\Delta\hat{x}_k = x_k - \hat{x}_k = x_k - \bar{x}_k - K\varepsilon_k = \Delta\bar{x}_k - K\varepsilon_k. \tag{11.98}$$

Post multiplication with $\varepsilon_k^T = (y_k - \bar{y}_k)^T = (D(x_k - \bar{x}_k) + w_k)^T$ gives

$$\begin{gathered} \Delta\hat{x}_k\varepsilon_k^T = \\ (x_k - \hat{x}_k)((x_k - \bar{x}_k)^T D^T + w_k^T) = (x_k - \bar{x}_k)((x_k - \bar{x}_k)^T D^T + w_k^T) - K\varepsilon_k\varepsilon_k^T. \end{gathered} \tag{11.99}$$

Using the mean operator $\mathrm{E}(\cdot)$ on both sides of the equal sign in (11.99) gives

$$0 = \bar{X}D^T - K\mathrm{E}(\varepsilon_k\varepsilon_k^T), \tag{11.100}$$

because

$$\begin{aligned} \mathrm{E}((x_k - \hat{x}_k)\varepsilon_k^T &= 0, && (11.101) \\ \mathrm{E}((x_k - \hat{x}_k)w_k^T) &= 0, && (11.102) \\ \mathrm{E}((x_k - \bar{x}_k)w_k^T) &= 0, && (11.103) \end{aligned}$$

when we are using the optimal Kalman filter gain K.

An alternative derivation is as follows

$$\mathrm{E}(\Delta\hat{x}_k\varepsilon_k^T) = \mathrm{E}((\Delta\bar{x}_k - K\varepsilon_k)\varepsilon_k^T) = 0. \tag{11.104}$$

And from Eq. (11.104) we have

$$\begin{aligned}\mathrm{E}((\Delta\bar{x}_k - K\varepsilon_k)\varepsilon_k^T) &= \mathrm{E}(\Delta\bar{x}_k \overbrace{(\Delta\bar{x}_k^T D^T + w_k^T)}^{\varepsilon_k^T}) - K\mathrm{E}(\varepsilon_k\varepsilon_k^T) \\ &= \bar{X}D^T - K\mathrm{E}(\varepsilon_k\varepsilon_k^T) = 0, \end{aligned} \tag{11.105}$$

since $\mathrm{E}(\Delta\bar{x}_k w_k^T) = 0$.

We then get from (11.100) (or equivalently (11.105)) that the optimal Kalman filter gain matrix in the filter is given by

$$K = \bar{X}D^T(D\bar{X}D^T + W)^{-1}, \tag{11.106}$$

where we have used that

$$\mathrm{E}(\varepsilon_k\varepsilon_k^T) = D\bar{X}D^T + W. \tag{11.107}$$

Let us now compare (11.106) with the expression for $\tilde{K} = AK$ for the Kalman filter gain in the predictor given by Equation (11.75). As we see, the equations are consistent and the same when $R_{12} = 0$. However, (11.106) will be valid even when the process noise and the measurements noise are correlated, but we then have to take $\bar{X}$ given by (11.82).

Equation for computing $\hat{X}$

The updating equation (11.87) can be expressed as follows

$$\hat{x}_k = \bar{x}_k + K(y_k - \bar{y}_k) = (I - KD)\bar{x}_k + KDx_k + Kw_k. \tag{11.108}$$

We can then write the estimator error $x_k - \hat{x}_k$ as follows

$$\begin{aligned} x_k - \hat{x}_k &= x_k - ((I - KD)\bar{x}_k + KDx_k + Kw_k) \\ &= (I - KD)(x_k - \bar{x}_k) + Kw_k. \end{aligned} \tag{11.109}$$

This gives

$$\hat{X}_k = (I - KD)\bar{X}_k(I - KD)^T + KWK^T. \tag{11.110}$$

Equation for updating $\bar{X}_k$

We have earlier deduced the Riccati equation for computing $\bar{X}_k$ in connection with the Kalman filter on prediction and innovations form. Se Equations (11.80)-(11.82). By substituting the expression for $\hat{X}_k$ given by (11.90) into Equation (11.91) gives Equation (11.80). This proves Equation (11.91).

Notice that a simple derivation (when $R_{12} = 0$) is as follows. We have

$$\bar{X}_{k+1} = \mathrm{E}(\Delta\bar{x}_{k+1}\Delta\bar{x}_{k+1}^T), \tag{11.111}$$

Using that

$$\Delta\bar{x}_{k+1} = x_{k+1} - \bar{x}_{k+1} = Ax_k + v_k - A\hat{x}_k = A\Delta\hat{x}_k + v_k, \tag{11.112}$$

where we have used that $\bar{x}_{k+1} = A\hat{x}$ when $R_{12} = 0$ in (11.88). Hence we find from (11.112) that

$$\bar{X}_{k+1} = A\hat{X}_k A^T + V. \tag{11.113}$$

since $\mathrm{E}(\Delta\hat{x}v_k^T) = 0$.

11.6.4 Summary

It is important to note that for discrete time systems, we have two formulations of the Kalman filter, one Kalman filter on innovations or prediction form, and one Kalman filter on apriori-aposteriori form for filtering or optimal state estimation. The Kalman filter gain in the innovations form is denoted $\tilde{K}$ and the Kalman filter gain in the filter is denoted K.

The relationship is given by $\tilde{K} = AK$ when the process noise v_k and the measurements noise w_k are uncorrelated, i.e. when $R_{12} = 0$. When the process noise and the measurements noise are correlated then the Kalman filter gain in the innovations form (the predictor) is given by

$$\tilde{K}_k = (A\bar{X}_k D^T + R_{12})(D\bar{X}_k D^T + W)^{-1},$$

and the gain in the filter used to compute the aposteriori state estimate is given by

$$K_k = \bar{X}_k D^T (D\bar{X}_k D^T + W)^{-1}.$$

As we see, the relationship is particularly simple and given by $\tilde{K}_k = AK_k$ when the noise are uncorrelated, i.e. when $R_{12} = 0$.

Chapter 12

The Kalman filter algorithm for discrete time systems

12.1 Contunuous time state space model

A continuous time nonlinear state space model can usually be written as

$$\dot{x} = f(x, u, v) \tag{12.1}$$
$$y = g(x, u) + w \tag{12.2}$$

where x is the state vector, u is the vector of known deterministic inputs, v is a process noise vector, w is a zero mean measurements noise vector, and y is a vector of measurements (observations).

This model is both driven by known deterministic inputs (u) and usually unknown process and measurements disturbances, (v and w).

12.2 Discrete time state space model

We will in this section formulate a discrete process model which can be used to design an Extended and possibly Augmented Kalman filter.

A discrete time model, which can be a discrete version of the continuous model, can usually be written as follows.

$$x_{t+1} = f_t(x_t, u_t, v_t) + dx_t \tag{12.3}$$
$$y_t = g_t(x_t, u_t) + w_t \tag{12.4}$$

where w_t is zero mean discrete measurements noise, dx_t is a zero mean process noise vector which also can represent unmodeled effects or uncertainity. The effect of adding the noise vector dx_t to the right hand side of the process noise is that it usually gives more tuning parameters in the process noise covariance matrix, which can result in a Kalman filter gain matrix with better properties of estimating the states.

We will next write this model on a form which is more convenient for nonlinear filtering (Extended Kalman filter, Jazwinski (1970)). The problem is the case when the process model function $f_t(\cdot)$ is a non-linear function of the pocess noise vector v_t. Assume that the statistical properties of v_t is known. In general, the statistical properties of the non linear function $f_t(v_t)$ is unknown. The idea is to augment a model for v_t with the process model such that the augmented model is linear in the process noise.

Assume the case when the process noise have known mean (or trend) $\bar{v}_t$ and that the noise can be modeled as

$$v_t = \bar{v}_t + dv_t \tag{12.5}$$

where dv_t is a zero mean white noise vector. The known mean process noise vector or trend $\bar{v}_t$ can be augmented into the vector of known deterministic inputs (u_t). The resulting model is then driven by both deterministic inputs (u_t and $\bar{v}_t$) and zero mean white process and measurements noise (dv_t and w_t). $f_t(\cdot)$ can in some cases be assumed to be a linear function of the white process noise vector (dv_t).

Assume next the better case when the process noise v_t can be modeled as a random walk (or drift), i.e.

$$v_{t+1} = v_t + dv_t \tag{12.6}$$

The vector v_t can be augmented into the state vector x_t. The resulting augmented model is linear in the process noise (dv_t).

The process model to be used in the filter is assumed to be of the following form, (i.e. linear in the process noise vector)

$$x_{t+1} = f_t(x_t, u_t) + \Omega_t v_t \tag{12.7}$$
$$y_t = g_t(x_t, u_t) + w_t \tag{12.8}$$

which is linear in terms of the unknown process and measurement white noise processes v_t and w_t, respectively. The input vector u_t is a collection of all (deterministic) known variables, including possibly measured process noise variables and manipulable process input variables. The system vector x_t can be an augmented vector of system states, possibly states in a process noise model and states in a parameter model, e.g. random walk (or drift) models.

Furthermore, the following statistical properties are assumed

$$\begin{array}{ll} \mathrm{E}(v_t) = 0 \text{ and } \mathrm{E}(v_t v_j^T) & = V\delta_{tj} \\ \mathrm{E}(w_t) = 0 \text{ and } \mathrm{E}(w_t w_j^T) & = W\delta_{tj} \end{array} \quad \text{where} \quad \delta_{tj} = \begin{cases} 1 \text{ if } j = t \\ 0 \text{ if } j \neq t \end{cases} \tag{12.9}$$

The linearized discrete time state space model is defined as

$$dx_{t+1} = \Phi_t dx_t + \Delta_t du_t + \Omega_t dv_t \tag{12.10}$$
$$dy_t = D_t dx_t + E du_t + w_t \tag{12.11}$$

where dx_t, du_t, dv_t and dy_t are deviations around some vectors of variables.

12.3 The Kalman filter algorithm

The algorithm presented is a formulation of the Extended and possibly Augmented Kalman filter. The algorithm is formulated, step for step, such that it can be directly implemented in a computer.

Algorithm 12.3.1 (Extended Kalman filter algorithm)
Step 0. *Initial values.*
Specify the apriori state vector, $\bar{x}_t$, and the apriori state covariance matrix, $\bar{X}_t$. ($\bar{x}_t$ and $\bar{X}_t$ are usually given from the previous sample of this algorithm. Note that t is discrete time.)

Step 1. *Measurements model uppdate.*

$$\bar{y}_t = g_t(\bar{x}_t, u_t) \tag{12.12}$$

Step 2. *The Kalman filter gain matrix.*
Linearized measurements model matrix

$$D_t = \left.\frac{\partial g_t(x_t, u_t)}{\partial x_t}\right|_{\bar{x}_t, u_t} \tag{12.13}$$

Kalman filter gain matrix.

$$K_t = \bar{X}_t D_t^T (D_t \bar{X}_t D_t^T + W)^{-1} \tag{12.14}$$

Step 3. *Aposteriori state estimate.*

$$\hat{x}_t = \bar{x}_t + K_t(y_t - \bar{y}_t) \tag{12.15}$$

Step 4. *Apriori state uppdate.*

$$\bar{x}_{t+1} = f_t(\hat{x}_t, u_t) \tag{12.16}$$

Define the state transition and the disturbance input matrices.

$$\Phi_t = \left.\frac{\partial f_t(x_t, u_t) + \Omega_t v_t}{\partial x_t}\right|_{\hat{x}_t, u_t} \tag{12.17}$$

$$\Omega_t = \left.\frac{\partial f_t(x_t, u_t) + \Omega_t v_t}{\partial v}\right|_{\hat{x}_t, u_t} \tag{12.18}$$

Step 5. *State covariance matrices.*
Aposteriori state covariance matrix.

$$\hat{X}_t = (I - K_t D_t)\bar{X}_t (I - K_t D_t)^T + K_t W K_t^T \tag{12.19}$$

Apriori state covariance matrix uppdate.

$$\bar{X}_{t+1} = \Phi_t \hat{X}_t \Phi_t^T + \Omega_t V \Omega_t^T \tag{12.20}$$

$\triangle$

Note that the matrix equation for the aposteriori state covariance matrix, Equation (12.19), is called the stabilized implementation, because it have better numerical properties than the other frequently used equations for $\hat{X}$, e.g.

$$\hat{X}_t = \bar{X}_t - \bar{X}_t D_t^T (D_t \bar{X}_t D_t^T + W)^{-1} D_t \bar{X}_t \tag{12.21}$$

$$\hat{X}_t = (I - K_t D_t) \bar{X}_t \tag{12.22}$$

The Algorithm 12.3.1 is all that is needed for the design of an Kalman filter application. Se also the next sections for pure details about implementation. However, for extreme accuracy of the computational results the (square root) algorithm by Bierman (1974) should be implemented

12.3.1 Example: parameter estimation

Assume the linear (measurement) equation

$$y_t = E_t u_t + w_t \tag{12.23}$$

where $y_t \in \Re^m$ and $u_t \in \Re^r$ are known. The error $w_t \in \Re^m$ is assumed to be a zero mean white noise process. $E_t \in \Re^{m \times r}$ is a matrix of unknown parameters. The problem adressed in this section is to estimate the (gain) matrix E_t.

We will first write the model into a more convenient form for parameter estimation. We have

$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix}_t = \begin{bmatrix} e_1^T \\ e_2^T \\ \vdots \\ e_m^T \end{bmatrix}_t u_t = \begin{bmatrix} u^T e_1 \\ u^T e_2 \\ \vdots \\ u^T e_m \end{bmatrix}_t = \begin{bmatrix} u_t^T & 0 & \cdots & 0 \\ 0 & u_t^T & \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & u_t^T \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_m \end{bmatrix}_t \tag{12.24}$$

which can be written as

$$y_t = \varphi_t^T \theta_t \tag{12.25}$$

where $y_t \in \Re^m$ is a vector of observations, $\varphi_t^T \in \Re^{m \times r \cdot m}$ is a matrix of (regression) known variables and $\theta_t \in \Re^{r \cdot m}$ is a vector of unknown parameters.

Hence, the parameter vector θ_t is formed from the rows in the matrix E and the matrix φ_t^T is a matrix with the known (input) vector u_t^T on the "diagonal". Note that in the Multiple Input Single Output (MISO) case, we simply have $\varphi_t^T = u_t^T$ and $\theta_t = E^T$.

Assume that the parameter vector θ_t is slowly varying. A reasonable model is then a so called random walk (or drift), i.e.

$$\theta_{t+1} = \theta_t + v_t \tag{12.26}$$

where v_t is a zero mean white noise process.

Problem

Use the Kalman filter Alorithm 12.3.1 to write an algorithm for parameter estimation based on the models given by Equations (12.25) and (12.26). Express the parameter estimates in terms of the apriori parameter estimate vector, i.e. $\bar{\theta}_t$.

12.4 Implementation

The Kalman filter matrix equations that are computed at each sample (if required) is given by,

1. Stabilized Kalman measurement uppdate equations.

$$K = XD^T(DXD^T + W)^{-1} \tag{12.27}$$
$$\hat{X} = (I - KD)X(I - KD)^T + KWK^T \tag{12.28}$$

2. Time uppdate apriori covariance matrix equation.

$$X = \Phi\hat{X}\Phi^T + V \tag{12.29}$$

where for simplicity $X := \bar{X}$.

We will in what follows count the number of multiplications which is required for one sample of the actual implementation and then suggest efficient implementations of the algorithm where the number of multiplications is considerably reduced.

The stabilized Kalman measurement uppdate Equation (12.28) is implemented in the following steps. The resulting matrix dimension and the number of multiplications required is identified to the right of each equations.

Algorithm 12.4.1 ("Bulk" implementation)

$$\begin{array}{lllll} WORK1 & = I - KD & (n \times n) & n^2m \\ WORK2 & = X\ WORK1^T & (n \times n) & n^3 \\ WORK3 & = WORK1\ WORK2 & (n \times n) & n^3 \\ X & = WORK3 + KWK^T & (n \times n) & 2n^2m \\ & & Total & 2n^3 + 3n^2m \end{array} \tag{12.30}$$

△

The total number of multiplications for Equation (12.28) is then given by

$$2n^3 + 3n^2m \quad (= 400 \text{ for } n = 5 \text{ and } m = 2) \tag{12.31}$$

The term KWK^T can be implemented more effectively as follows

$$\begin{array}{llll} WORK1 & = KW & (n \times m) & nm \\ WORK2 & = WORK1\ K^T & (n \times n) & n^2m \end{array} \tag{12.32}$$

The total number of multiplications is in this case given by

$$2n^3 + 2n^2m + nm \quad (= 360 \text{ for } n = 5 \text{ and } m = 2) \tag{12.33}$$

Multiplications can be saved if the symmetry of the matrix terms $(I - KD)X(I - KD)^T$ and KWK^T are utilized. Only the lower or upper part of the latter terms needs to be computed.

Algorithm 12.4.2 (Computations of symmetrical parts only)

$$\begin{array}{lll} WORK1 = I - KD & (n \times n) & n^2 m \\ WORK2 = X\ WORK1^T & (n \times n) & n^3 \\ WORK3 = WORK1\ WORK2 & (n \times n) & n\frac{n(n+1)}{2} \\ WORK1 = K\ W & (n \times m) & nm \\ X \quad = WORK3 + WORK1\ K^T & (n \times n) & m\frac{n(n+1)}{2} \\ & \mathit{Total} & \frac{3}{2}n^3 + \frac{3}{2}n^2 m + \frac{1}{2}n^2 + \frac{3}{2}nm \end{array} \tag{12.34}$$

△

The total number of multiplications is in this case given by

$$\frac{3}{2}n^3 + \frac{3}{2}n^2 m + \frac{1}{2}n^2 + \frac{3}{2}nm \quad (= 290 \text{ for } n = 5 \text{ and } m = 2) \tag{12.35}$$

In general, the most efficient implementation of Equation (12.28) with respect to the number of multiplications is probably as follows. However, both algorithms (12.4.1) and (12.4.2) are probably better conditioned with respect to positive definiteness of the computed covariance matrix.

Algorithm 12.4.3 (Biermans implementation)

$$\begin{array}{llll} WORK1 = XD^T & (n \times m) & n^2 m & \\ X \quad = X - K\ WORK1^T & (n \times n) & n^2 m & \\ WORK2 = KW & (n \times m) & nm & \\ WORK1 = XD^T - WORK2 & (n \times m) & n^2 m & \\ X \quad = X - WORK1\ K^T & (n \times n) & n^2 m & m\frac{n(n+1)}{2} \\ & \mathit{Total} & (4n^2 + n)m & (\frac{5}{2}n^2 + \frac{3}{2}n)m \end{array} \tag{12.36}$$

△

Note that the matrix product XD^T used initially in Algorithm 12.4.3 is available from the computation of the gain matrix K. Therefore the total number of multiplications by Algorithm 12.4.3 can be reduced by $n^2 m$ for comparison with Algorithms 12.4.1 and 12.4.2. The total number of multiplications required to form the a posteriori state covariance matrix $\hat{X}$ is illustrated in the following table.

Table 1: Comparison of number of multiplications for $m = 2$

Algorithm	Total	N = 3	N = 5	Remarks
4.1	$2n^3 + 3n^2 m$	108	400	
4.2	$\frac{3}{2}n^3 + \frac{3}{2}n^2 m + \frac{1}{2}n^2 + \frac{3}{2}nm$	81	290	(12.37)
4.3	$(3n^2 + n)m$	64	160	
4.3 Symmetrized	$(\frac{5}{2}n^2 + \frac{3}{2}n)m$	54	140	

The a priori state covariance uppdate matrix Equation (12.29) can be directly implemented with $2n^3$ multiplications or with $n^3 + n\frac{n(n+1)}{2} = \frac{3}{2}n^3 + \frac{1}{2}n^2$ if the symmetry of the resulting product $\Phi\hat{X}\Phi^T$ is utilized.

Note that the structure of the Φ matrix should be utilized if it is sparse. For the $N = 5$ and $M = 2$ example given in this note, only 36 multiplications are needed to form $\bar{X}$ compared to 250 (or 200 if symmetry is utilized) in the general case.

Skogn implementation: $72 + 400 + 250 = 722$.

Symetrical implementation: $67 + 290 + 200 = 557$.

Symetrical and structure: $67 + 290 + 36 = 393$.

4.3 symmetrized and structure: $67 + 140 + 36 = 243$.

References

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Chapter 13

Robustness in LQ and LQG systems

13.1 Return difference equation

The Riccati equation can be formulated in the frequency domain through the so called return difference equation. This equation is of central importance in connection with robustness properties of the LQ controller.

Theorem 13.1.1 (return difference equation)
The Riccati equation can be written as

$$[I+H_0(-s)]^T P[I+H(s)] = P + H_p^T(-s)QH_p(s) \tag{13.1}$$

where

$$G = -P^{-1}B^T R, \tag{13.2}$$
$$H_0(s) = -G(sI-A)^{-1}B, \tag{13.3}$$
$$H_p(s) = (sI-A)^{-1}B. \tag{13.4}$$

Proof 13.1 *The proof is divided into two parts.*

Part 1 *The return difference equation can be written as*

$$\begin{gathered} P - PG(sI-A)^{-1}B - B^T(-sI-A^T)^{-1}G^T P \\ +B^T(-sI-A^T)^{-1}G^T PG(sI-A)^{-1}B = P + B^T(-sI-A^T)^{-1}Q(sI-A)^{-1}B \end{gathered}$$

and

$$\begin{gathered} -PG(sI-A)^{-1}B - B^T(-sI-A^T)^{-1}G^T \\ +B^T(-sI-A^T)^{-1}G^T PG(sI-A)^{-1}B = B^T(-sI-A^T)^{-1}Q(sI-A)^{-1}B \end{gathered} \tag{13.5}$$

Part 2 *Hence, we have to prove (13.5). The algebraic Riccati equation $A^TR+RA-RBP^{-1}B^TR+Q$ can be written as*

$$-A^TR - RA + G^T PG = Q \tag{13.6}$$

where we have used that $G = -P^{-1}B^TR$ is the optimal state feedback matrix. Adding sR and subtracting $-sR$ to the left hand side gives

$$(-sI-A^T)R + R(sI-A) + G^T PG = Q \tag{13.7}$$

Pre-multiplication with $B^T(-sI-A^T)^{-1}$ and post-multiplication with $(sI-A)^{-1}B$ gives

$$\begin{gathered} B^TR(sI-A)^{-1}B + B^T(-sI-A^T)^{-1}RB + B^T(-sI-A^T)^{-1}G^T PG(sI-A)^{-1}B \\ = B^T(-sI-A^T)^{-1}Q(sI-A)^{-1}B \end{gathered} \tag{13.8}$$

From $G = -P^{-1}B^TR$ we have that $B^TR = -PG$. This gives

$$\begin{gathered} -PG(sI-A)^{-1}B - B^T(-sI-A^T)^{-1}G^T P + B^T(-sI-A^T)^{-1}G^T PG(sI-A)^{-1}B \\ = B^T(-sI-A^T)^{-1}Q(sI-A)^{-1}B \end{gathered} \tag{13.9}$$

which is equivalent to (13.5). QED

13.2 Robustness of LQ systems

Consider a single input LQ system. From the *return difference equation* we have that

$$|1 + h_0| \geq 1 \tag{13.10}$$

where the loop transfer function is $h_0 = -G(sI - A)^{-1}B$ and $G = -P^{-1}B^T R$ and R is the positive solution to the ARE.

The inequality (13.10) implies that the curve $h_0(j\omega)$ does not enter a circle with center $(-1, 0)$ and radius $r \geq 1$ in the complex plane. This can be shown by using that $h_0(j\omega) = \Re e h_0 + j\Im m h_0$, which gives the circle equation $(\Re e h_0 + 1)^2 + \Im m h_0^2 = r^2$ where the radius satisfy $r^2 \geq 1$.

Considder the possibile values of $h_0(j\omega)$ along the real axis. From the inequality (13.10) we have $-(1 + h_0) \geq 1$ which gives $h_0 \leq -2$ and from $(1 + h_0) \geq 1$ we have that $h_0 \geq 0$. Considder that there is a multiplicative uncertainity in the system, which results in a perturbed loop transfer function $h = kh_0$ where k is a constant uncertainity parameter. The perturbed system is on the stability limit if $|h(j\omega)| = 1$ and $\angle h(j\omega) = -180°$. This gives that $k = \frac{1}{|h_0|}$.

13.2.1 Gain margin

From the above we have the condition $h_0 \geq 0$ which gives $k \leq \infty$ or $k = \infty$ if only the negative real axis is considered. Recall that the gain margin (GM) is the factor by which the loop gain may be increased before the closed loop system becomes unstable. hence, we have a gain margin

$$GM = k = \infty \tag{13.11}$$

13.2.2 Gain reduction margin

The condition $h_0 \leq -2$ gives $k \leq \frac{1}{2}$. Hence, the loop gain may be reduced by a factor

$$0 \leq k \leq \frac{1}{2} \tag{13.12}$$

before the system becomes unstable. This is defined as the *Gain reduction margin* property of the LQ regulator.

Example 13.1 (Gain margin with LQ regulator)
Considder that we have a model

$$\dot{x}_m = x_m + u, \tag{13.13}$$
$$y_m = x_m, \tag{13.14}$$

for a real plant

$$\dot{x} = x + mu, \tag{13.15}$$
$$y = x. \tag{13.16}$$

The difference between the plant and the model is only the parameter m. Considder now that an LQ regulator is designed based on the model (13.13) and (13.14) and applied to the plant (13.15) and (13.16). The problem which is adressed is now to find out how large perturbations in the parameter m we can tolerate before the system becomes unstable.

The LQ performance index is

$$J = \int_0^\infty (qy^T y + pu^2)dt. \tag{13.17}$$

The solution to the algebraic Riccati equation $2ar - \frac{b^2}{p}r^2 + q = 0$ and the optimal state feedback are

$$r = p(1 + \sqrt{1 + \frac{q}{p}}), \tag{13.18}$$

$$g = -\frac{1}{p}r = -(1 + \sqrt{1 + \frac{q}{p}}). \tag{13.19}$$

The control to the plant is chosen as $u = gx$. The closed loop system is then described by $\dot{x} = (a + mg)x$. The eigenvalue of the closed loop system is $\lambda = a + mg$ and for stability we must have that

$$\lambda = 1 + mg = 1 - m(1 + \sqrt{1 + \frac{q}{p}}) \leq 0. \tag{13.20}$$

This gives that

$$\frac{1}{1 + \sqrt{1 + \frac{q}{p}}} \leq m. \tag{13.21}$$

Considder now the two cases $\frac{q}{p} = 0$ and $\frac{q}{p} \to \infty$.

$$\begin{array}{ll} \frac{q}{p} = 0 & \Rightarrow \frac{1}{2} \leq m \leq \infty \\ \frac{q}{p} \to \infty & \Rightarrow 0 \leq m \leq \infty \end{array} \tag{13.22}$$

This means that the LQ system is guaranteed to be stable if

$$\frac{1}{2} \leq m \leq \infty \tag{13.23}$$

irrespective of the choice of weight parameters $q \geq 0$ and $p > 0$.

13.3 Robustness of LQG systems

The results in the paper by Doyle (1978), with title *Guaranteed Margins for LQG regulators* and abstract *There are none* are reviewed and worked out in the following example.

Example 13.2 (LQG example, Doyle (1978).)
Consider that we have a model

$$\dot{x} = \overbrace{\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}}^{A} x + \overbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}^{B} u + \overbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}^{C} v \tag{13.24}$$

$$y = \overbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}^{D} x + w \tag{13.25}$$

where $x = \begin{bmatrix} x_1 & x_2 \end{bmatrix}^T$ *is the state,* v *and* w *is Gaussian white noise with variance* $E(v^2) = V = \sigma > 0$ *and* $E(w^2) = W = 1$*, for the (real) process*

$$\dot{x} = Ax + B_p u + Cv \tag{13.26}$$

$$y = Dx + w \tag{13.27}$$

where

$$B_p = \begin{bmatrix} 0 \\ m \end{bmatrix}, \tag{13.28}$$

and where m *is an unknown parameter, but assumed to be close to* $m = 1$*.*

An infinite horizon LQ controller and a Kalman filter are constructed based on the process model (13.24) and (13.25), and applied to the plant (13.26) and (13.27).

Let the LQ performance index by

$$J = \int_0^\infty (qy^T y + u^T P u)dt = \int_0^\infty (x^T Q x + u^T P u)dt, \tag{13.29}$$

where

$$Q = qDD^T = q \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \; q > 0, \tag{13.30}$$

and $P = 1$*. The LQ controller minimizing (13.29) is given by*

$$u = G\hat{x}, \tag{13.31}$$

where

$$G = \begin{bmatrix} -f & -f \end{bmatrix}, \tag{13.32}$$

and where

$$f = 2 + \sqrt{4 + q}. \tag{13.33}$$

The Kalman filter is

$$\dot{\hat{x}} = A\hat{x} + Bu + K(y - Dx),. \tag{13.34}$$

where the Kalman filter gain is

$$K = \begin{bmatrix} d \\ d \end{bmatrix}, \tag{13.35}$$

and where

$$d = 2 + \sqrt{4+\sigma}. \tag{13.36}$$

The closed loop system, determined by applying the control (13.31) and (13.34) to the plant (13.26) and (13.27) is given by

$$\begin{bmatrix} \dot{x} \\ \dot{\hat{x}} \end{bmatrix} = \overbrace{\begin{bmatrix} A & B_pG \\ KD & A+BG-KD \end{bmatrix}}^{A_{cl}} \begin{bmatrix} x \\ \hat{x} \end{bmatrix}, \tag{13.37}$$

with

$$A_{cl} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & -mf & -mf \\ d & 0 & 1-d & 1 \\ d & 0 & -d-f & 1-f \end{bmatrix}. \tag{13.38}$$

The stability of the LQG system is defined by the eigenvalues of matrix A_{cl}. The characteristic polynomial is (use e.g. MAPLE to show this)

$$\mid \lambda I - A_{cl} \mid = \lambda^4 + c_3\lambda^3 + c_2\lambda^2 + c_1\lambda + c_0, \tag{13.39}$$

where the polynomial coefficients are

$$c_0 = 1 + (1-m)df, \tag{13.40}$$
$$c_1 = d + f - 4 + 2(m-1)df, \tag{13.41}$$
$$c_2 = df - 2f - 2d + 6, \tag{13.42}$$
$$c_3 = f + d - 4. \tag{13.43}$$

From Rouths stability criterion we have that a necessary (but not sufficient) condition for stability is that all coefficients $(1, c_3, c_2, c_1, c_0)$ in the characteristic equation is positive. The nominal LQG system with $m = 1$ is stable so we know that $c_3 > 0$ and $c_2 > 0$. Note also that only c_1 and c_0 is dependent upon the unknown parameter m.

A necessary condition for stability is then that

$$c_0 = 1 + (1-m)df > 0, \tag{13.44}$$
$$c_1 = d + f - 4 + 2(m-1)df > 0. \tag{13.45}$$

Obviously, this is true for $m = 1$. This is also true if

$$m_{low} < m < m_{upp}. \tag{13.46}$$

where

$$m_{low} = 1 - \Delta m_{low}, \;\; \Delta m_{low} = \frac{d+f-4}{2df} = \frac{\sqrt{4+q} + \sqrt{4+\sigma}}{2df},. \tag{13.47}$$

$$m_{upp} = 1 + \Delta m_{upp}, \;\; \Delta m_{upp} = \frac{1}{df}.. \tag{13.48}$$

Note that both d and f are positive and that $d + f - 4 = \sqrt{4+q} + \sqrt{4+\sigma} > 0$. Hence, there exist $m \neq 1$ for which the necessary conditions $c_0 > 0$ and $c_1 > 0$ are satisfied, We have assumed that q and σ are finite. However, the problem is that $m_{low} \to 1$ and $m_{upp} \to 1$ when $q \to \infty$ and/or $\sigma \to \infty$. This means that the margins (Δm_{low} and Δm_{upp}) can be made arbitrarily small for sufficiently large parameters q and σ. Note that $\Delta m_{low} \to 0$ and $\Delta m_{upp} \to 0$ when $q \to \infty$ and/or $\sigma \to \infty$.

Consider a particular LQG design with parameters $q = \sigma = 12$. The necessary conditions for stability are in this case satisfied if

$$0.889 < m < 1.027. \tag{13.49}$$

It can be shown numerically that

$$0.9105 < m < 1.027 \tag{13.50}$$

is both necessary and sufficient for stability of the particular LQG system.

Hence the margins should be checked for each specific LQG design

13.4 Exercises

Exercise 13.1 (Gain margin in LQ system) *Considder a SISO plant with one state and model parameters $A = -1$, $B = 1$. Assume that we have an multiplicative uncertainity in the real plant input matrix. I.e. the real plant is $\dot{x} = Ax + B_p u$ with $B_p = mB$ where m is the multiplicative uncertainity. Show that the closed loop LQ system have gain margin*

$$-\frac{1}{\sqrt{1+\frac{q}{p}} - 1} \leq m. \tag{13.51}$$

Part IV

PREDICTIVE CONTROL

Chapter 14

Introduction

Chapter 15

Model predictive control

Chapter 16

Unconstrained and constrained optimization

Chapter 17

Introductory examples

Chapter 18

Extension of the control objective

Chapter 19

DYCOPS5 paper: On model based predictive control

Chapter 20

Extended state space model based predictive control

Chapter 21

Constraints for Model Predictive Control

Chapter 22

More on constraints and Model Predictive Control

Chapter 23

EMPC: The case with a direct feed trough term in the output equation

Part V

NONLINEAR CONTROL

Chapter 24

Eksempel på bruk av ulineær dekobling

Example 24.1 (Regulering av ulineært SISO system)
Gitt en prosess beskrevet/modellert med

$$\dot{x} = f(x, u) \tag{24.1}$$

der

$$f(x, u) = -\frac{u}{(x+1)^2}. \tag{24.2}$$

Vi innfører nå et ekvivalent pådrag $\tilde{u}$ *slik at*

$$\dot{x} = f(x, u) = \tilde{u}. \tag{24.3}$$

Dette betyr at prosessen er en ren integrator sett fra det ekvivalente pådraget $\tilde{u}$*. Prosessens pådrag* u *kan nå bestemmes ved å løse* $f(x, u) = \tilde{u}$ *med hensyn på* u*. Dvs. vi løser*

$$-\frac{u}{(x+1)^2} = \tilde{u} \tag{24.4}$$

mht u *som gir*

$$u = -(x+1)^2 \tilde{u}. \tag{24.5}$$

Ligning (24.5) er å betrakte som en kompensator som plasseres før prosessen. Det vil være tilstrekkelig med en proporsjonal-regulator for å generere det ekvivalente pådraget $\tilde{u}$ *og for å regulere prosessen* $\dot{x} = \tilde{u}$*. Dvs.*

$$\tilde{u} = K_p(x_0 - x) \tag{24.6}$$

der x_0 *er et spesifisert settpunkt og der* K_p *er en konstant. Vi ser forøvrig at vi må kreve at* $K_P > 0$ *for at det lukkede systemet skal være stabilt.* K_p *kan for eksempel velges slik at man får en spesifisert tidskonstant* $T = \frac{1}{K_p}$ *etter en settpunktsendring i* x_0*.*

Example 24.2 (Regulering av ulineært 2×2 system)
Anta at en reaksjon

$$sA \overset{k}{\to} B \tag{24.7}$$

foregår i en isoterm tank med ideell omrøring der $k = 1$ er reaksjons hastighets konstant fra stoff A til stoff B og $s = 2$.

Definer u_1 som massestrømen inn til rektoren og u_2 som sammensetningen av stoff A i u_1. Likeledes defineres x_1 som sammensetningen av stoff A i reaktoren og x_2 som sammensetningen av stoff B i reaktoren. Prosessen og reaksjonen er kontinuerlig, dvs. at det er en kontinuerlig gjennomstrømning i reaktoren.

En modell for prosessen kan bestemmes på følgende måte. Vi setter opp komponent massebalanser for stoffene A og B.

$$\frac{d}{dt}(Vx_1) = u_1u_2 - u_1x_1 - skVx_1^2, \tag{24.8}$$

$$\frac{d}{dt}(Vx_2) = -u_1x_2 + kVx_1^2. \tag{24.9}$$

der $V = 1$ er reaktorens tank volum som antas konstant. Dette kan videre skrives slik

$$\dot{x}_1 = \frac{u_1}{V}(u_2 - x_1) - skx_1^2, \tag{24.10}$$

$$\dot{x}_2 = -\frac{u_1}{V}x_2 + kx_1^2. \tag{24.11}$$

Denne prosessen er beskrevet i Fjeld (1971) s. 32. men uten utledning.

Vi innfører ekvivalente pådrag $\tilde{u}_1$ og $\tilde{u}_2$ slik at

$$\dot{x}_1 = \tilde{u}_1, \tag{24.12}$$

$$\dot{x}_2 = \tilde{u}_2. \tag{24.13}$$

Dette betyr at prosessens pådrag kan bestemmes ved å løse

$$\frac{u_1}{V}(u_2 - x_1) - skx_1^2 = \tilde{u}_1 \tag{24.14}$$

$$-\frac{u_1}{V}x_2 + kx_1^2 = \tilde{u}_2 \tag{24.15}$$

med hensyn på u_1 og u_2. Dette gir

$$u_1 = -\frac{V}{x_2}(\tilde{u}_2 - kx_1^2) \tag{24.16}$$

$$u_2 = x_1 + \frac{V}{u_1}(\tilde{u}_1 + skx_1^2) \tag{24.17}$$

Reguleringssløyfen kan nå lukkes med for eksempel to enkeltsløyfe proporsjonal-regulatorer (PI eller PID regulatorer kan også benyttes). Dvs.

$$\tilde{u}_1 = K_{p1}(x_{10} - x_1) \tag{24.18}$$

$$\tilde{u}_2 = K_{p2}(x_{20} - x_1) \tag{24.19}$$

der x_{10} *og* x_{20} *er spesifiserte settpunkt og* K_{p1} *og* K_{p2} *er positive konstanter.*

For bruk ved analyse og simulering så vil vi nå presentere stasjonærverdiene til reaktormodellen (24.10) og (24.11). Fra (24.10) har vi at

$$\dot{x}_1^s = \frac{u_1^s}{V}(u_2^s - x_1^s) - sk(x_1^s)^2 = 0, \tag{24.20}$$

$$\dot{x}_2^s = -\frac{u_1^s}{V}x_2^s + k(x_1^s)^2 = 0. \tag{24.21}$$

dette gir

$$x_1^s = \frac{-u_1^s + \sqrt{(u_1^s)^2 + 4skVu_1^su_2^s}}{2skV}, \tag{24.22}$$

$$x_2^s = \frac{kV(x_1^s)^2}{u_1^s}. \tag{24.23}$$

Dersom de stasjonære pådragene er gitt ved $u_1^s = 10$ *og* $u_2^s = 1$ *har vi at de stasjonære tilstandene er gitt ved* $x_1^s = 0.8541$ *og* $x_2^s = 0.0729$*. Det er disse stasjonærverdiene som er benyttet ved simulering av reaktorreguleringssystemet.*

Simuleringsresultater for prosessen regulert med ulineær dekobling er vist i figur 24.1. For å kunne sammenligne viser vi simuleringsresultater for samme prosess regulert med to enkeltsløyfe PI regulatorer i figur 24.2.

Av figur 24.1 ser vi at responsene i x_1 *og* x_2 *er dekoblet. Det vil for eksempel si at et settpunktsendring i* x_{10} *ikke har innvirkning på responsen i* x_2*. Dette er ikke tilfellet dersom prosessen reguleres med to enkeltsløyfe PI regulatorer som vist i figur 24.2.*

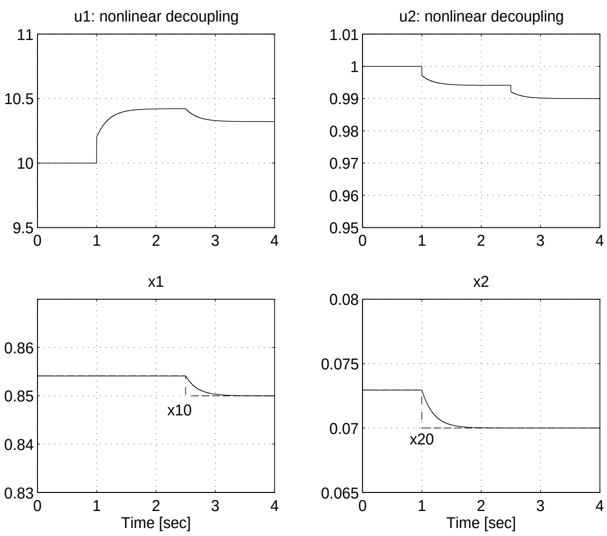

Figure 24.1: Figuren viser simulering av prosessen i eksempel 24.2 regulert med ulineær dekobling. Settpunktene x_{10} og x_{20} er stiplet. P regulatorene i (24.18) og (24.19) har parametre $K_{p1} = K_{p2} = 5$. Figuren er generert av MATLAB scriptet **nl_ex2.m**.

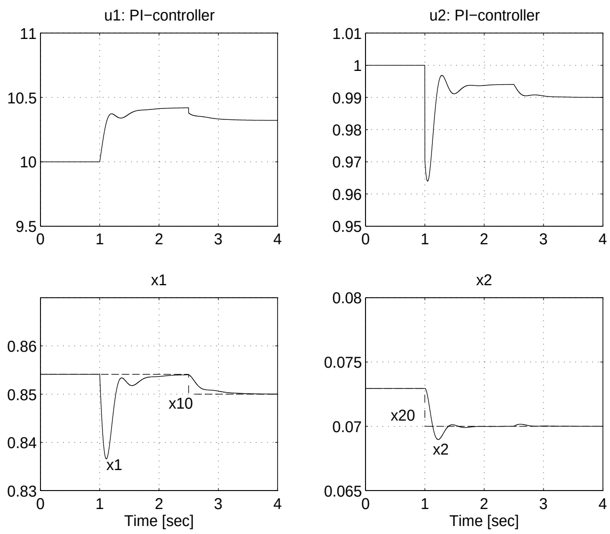

Figure 24.2: Figuren viser simulering av prosessen i eksempel 24.2 regulert med to enkeltsløyfe PI regulatorer. Settpunktene x_{10} og x_{20} er stiplet. Begge PI regulatorene har parametre $K_p = 10$ og $T_i = 0.1$. Figuren er generert av MATLAB scriptet **nl_ex2_pi.m**.

Part VI

RECURSIVE SUBSPACE IDENTIFICATION

Chapter 25

Recursive identification

Chapter 26

Recursive implementation of a subspace identification algorithm: RDSR

Chapter 27

Additional exercises

Appendix A

Linear Algebra and Matrix Calculus

A.1 Trace of a matrix

The trace of a $n \times m$ matrix A is defined as the sum of the diagonal elements of the matrix, i.e.

$$tr(A) = \sum_{i=1}^{n} a_{ii} \tag{A.1}$$

We have the following trace operations on two matrices A and B of apropriate dimensions

$$tr(A^T) = tr(A) \tag{A.2}$$
$$tr(AB^T) = tr(A^T B) = tr(B^T A) = tr(BA^T) \tag{A.3}$$
$$tr(AB) = tr(BA) = tr(B^T A^T) = tr(A^T B^T) \tag{A.4}$$
$$tr(A \pm B) = tr(A) \pm tr(B) \tag{A.5}$$

A.2 Gradient matrices

$$\frac{\partial}{\partial X}\ tr[X] = I \tag{A.6}$$
$$\frac{\partial}{\partial X}\ tr[AX] = A^T \tag{A.7}$$
$$\frac{\partial}{\partial X}\ tr[AX^T] = A \tag{A.8}$$
$$\frac{\partial}{\partial X}\ tr[AXB] = A^T B^T \tag{A.9}$$
$$\frac{\partial}{\partial X}\ tr[AX^T B] = BA \tag{A.10}$$
$$\frac{\partial}{\partial X}\ tr[XX] = 2X^T \tag{A.11}$$
$$\frac{\partial}{\partial X}\ tr[XX^T] = 2X \tag{A.12}$$
$$\frac{\partial}{\partial X}\ tr[X^n] = n(X^{n-1})^T \tag{A.13}$$
$$\frac{\partial}{\partial X}\ tr[AXBX] = A^T X^T B^T + B^T X^T A^T \tag{A.14}$$

$$\frac{\partial}{\partial X}\ tr[e^{AXB}] = (Be^{AXB}A)^T \tag{A.15}$$

$$\frac{\partial}{\partial X} tr[XAX^T] = 2XA, \text{ if } A = A^T \tag{A.16}$$

$$\frac{\partial}{\partial X^T}\ tr[AX] = A \tag{A.17}$$
$$\frac{\partial}{\partial X^T}\ tr[AX^T] = A^T \tag{A.18}$$
$$\frac{\partial}{\partial X^T}\ tr[AXB] = BA \tag{A.19}$$
$$\frac{\partial}{\partial X^T}\ tr[AX^T B] = A^T B^T \tag{A.20}$$
$$\frac{\partial}{\partial X^T}\ tr[e^{AXB}] = Be^{AXB}A \tag{A.21}$$

A.3 Derivatives of vector and quadratic form

The derivative of a vector with respect to a vector is a matrix. We have the following identities:

$$\frac{\partial x}{\partial x^T} = I \tag{A.22}$$
$$\frac{\partial}{\partial x}\ (x^T Q) = Q \tag{A.23}$$
$$\frac{\partial}{\partial x}\ (Qx) = Q^T \tag{A.24}$$
$$\tag{A.25}$$

The derivative of a scalar with respect to a vector is a vector. We have the following identities:

$$\frac{\partial}{\partial x}\ (y^T x) = y \tag{A.26}$$
$$\frac{\partial}{\partial x}\ (x^T x) = 2x \tag{A.27}$$
$$\frac{\partial}{\partial x}\ (x^T Qx) = Qx + Q^T x \tag{A.28}$$
$$\frac{\partial}{\partial x}\ (y^T Qx) = Q^T y \tag{A.29}$$

Note that if Q is symmetric then

$$\frac{\partial}{\partial x}\ (x^T Qx) = Qx + Q^T x = 2Qx. \tag{A.30}$$

A.4 Matrix norms

The most frequently used matrix norm in numerical analysis and linear algebra is the Frobenius norm (the F-norm).

The trace of the matrix product $A^T A$ is related to the Frobenius norm of A as follows

$$\|A\|_F^2 = \text{tr}(A^T A), \tag{A.31}$$

where $A \in \mathbb{R}^{N\times m}$.

A frequently used notation and expression for the matrix Frobenius norm is also

$$\|A\|_F^2 = (\sum_{i=1}^{N}\sum_{j=1}^{m} a_{ij})^{\frac{1}{2}}, \tag{A.32}$$

i.e. equal the square root of the sum of all elements.

An important property of the Frobenius norm is that it is invariant with respect to orthogonal transformation. Assume given two orthogonal matrices Q and Z with appropriate dimensions we have that

$$\|A\|_F = \|QAZ\|_F. \tag{A.33}$$

A.5 Linearization

Given a vector function $f(x) \in \mathbb{R}^m$ where $x \in \mathbb{R}^n$. The derivative of the vector f with respect to the row vector x^T is defined as

$$\frac{\partial f}{\partial x^T} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \frac{\partial f_m}{\partial x_2} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix} \in \mathbb{R}^{m\times n} \tag{A.34}$$

Given a non-linear differentiable state space model

$$\dot{x} = f(x, u), \tag{A.35}$$
$$y = g(x). \tag{A.36}$$

A linearized model around the stationary points x_0 and u_0 is

$$\dot{\delta x} = Ax + Bu, \tag{A.37}$$
$$\delta y = Dx, \tag{A.38}$$

where

$$A = \frac{\partial f}{\partial x^T}\,|_{x_0,u_0}, \tag{A.39}$$
$$B = \frac{\partial f}{\partial u^T}\,|_{x_0,u_0}, \tag{A.40}$$
$$D = \frac{\partial g}{\partial x^T}\,|_{x_0,u_0}, \tag{A.41}$$

and where

$$x = x - x_0, \tag{A.42}$$
$$u = u - u_0. \tag{A.43}$$

A.6 Kronecer product matrices

Given a matrix $X \in \mathbb{R}^{N \times r}$. Let I_m be the $(m \times m)$ identity matrix. Then

$$(X \otimes I_m)^T = X^T \otimes I_m, \tag{A.44}$$

$$(I_m \otimes X)^T = I_m \otimes X^T. \tag{A.45}$$

References