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PI/PID controller tuning via LQR approach

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Abstract

This paper presents a new optimal PI/PID controller tuning algorithms for low-order plus time-delay processes via LQR approach. A new criterion for selection of the Q and R matrices is proposed which will lead to the desired natural frequency and damping ratio of the closed-loop system. The examples with various dynamics are included to demonstrate the effectiveness of the tuning algorithms and show significant improvement over the existing PID tuning methods. The robustness property of the tuning algorithms is also analyzed, and it is shown that the LQR system is robustly stable for the small modeling error.

Keywords: PI/PID controller; Closed-loop; Tuning algorithms; LQR

1. Introduction

Proportional-Integral-Derivative (PID) controller has remained as the most commonly used controllers in industrial process control for 50 yr even though great progress in control theory has been made over the period. This is because it has a simple structure and is easily understood by the control engineers (Luyben, 1990). As early as 1942, Ziegler and Nichols (1942) proposed the first PID tuning method and surprisingly it is still widely used in practice. However, high performance is always the design target in industrial control applications and the Ziegler–Nichols method is insufficient in such applications. Recently, many techniques have been reported to improve PID tuning. Among them are the refined Ziegler–Nichols method (Hang, Astrom & Ho, 1991); the gain-phase margin method (Astrom & Hagglund, 1988; Ho, Hang & Cao, 1995a; Ho, Hang & Zhou, 1995b); the optimization of cost function method (Shinskey, 1988; Zhuang & Atherton, 1993), IMC-based PID controller design (Morari & Zafiriou, 1989) and tuning of open-loop unstable processes (Rotstein & Lewin, 1991). The model-based tuning methods are very encouraging (Huang, Chen, Lai & Wang, 1996; Morari & Zafiriou, 1989).

Linear Quadratic Regulator (LQR) design technique is well known in modern optimal control theory and has been widely used in many applications (Lewis & Syrmos, 1995). It has a very nice robustness property, i.e., if the process is of single-input and single-output, then the control system has at least the phase margin of 60° and the gain margin of infinity. This attractive property appeals to the practicing engineers. Thus, the LQR theory has received considerable attention since 1950s. In the context of optimal PID tuning, typical performance indices are the integral of squared error and time weighted error. With this kind of performance criterions, the integral of squared error (squared time weighted error) is calculated using "Astrom's integral algorithm" recursively if the process transfer function is known (Zhuang & Atherton, 1993). The Pade approximation is used to replace the time delay and then obtain the optimal PID controller for a first-order plus time-delay process. It is however noted that the Pade approximation may be inadequate for large normalized time delay. It is also noticed that an analytical tuning formula cannot be obtained via this optimization. The computational procedure to minimize the performance criterion is complicated and thus unsuitable for on-line applications.

In this paper, the LQR approach is employed to develop an optimal PI/PID controller tuning algorithm for the low-order plus time-delay model. A new criterion for selection of the Q and R matrices is proposed which will lead to the desired natural frequency and damping

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ratio of the closed-loop system. The examples with various dynamics are included to demonstrate the effectiveness of the tuning algorithms and show significant improvement over some existing best PID tuning methods. Finally, the robustness property of the tuning algorithms is analyzed, and it is shown that the LQR system is robustly stable for the small modeling error.

The paper is organized as follows. In Section 2, the LQR solution for a linear process with time delay is presented. An optimal PI tuning algorithm for a first-order plus time-delay model is derived via the LQR approach in Section 3. Various simulations are given in Section 4. Section 5 proposes the optimal PID tuning algorithm. Section 6 considers the robustness issue of the PI tuning algorithm. Section 7 concludes the paper.

2. LQR solution for time-delay systems

Consider a linear process with time delay described by

$$\dot{x}(t) = Ax(t) + Bu(t-L) \tag{1}$$

and the control performance specification measured in terms of

$$J = \int_0^\infty (x^{\mathrm{T}}(t)Qx(t) + u^{\mathrm{T}}(t)Ru(t))\,\mathrm{d}t, \tag{2}$$

where A, B, C, Q and R are given matrices with proper dimensions, $Q \geqslant 0$ and $R > 0$, $u(t) = 0$, when $t < 0$. The LQR problem is to find the optimal control $u(t)$ such that J in Eq. (2) is minimized. We decompose the dynamic process (1) into two stages: (I) when $0 \leqslant t < L$, $u(t-L) = 0$, there is no input signal to process (1) so that

$$\dot{x}(t) = Ax(t), \quad 0 \leqslant t < L, \tag{3}$$

(II) when $t \geqslant L$, the process has a possible non-zero input signal. In this stage, let $\hat{u}(t) = u(t-L)$, $t \geqslant L$, we have

$$\dot{x}(t) = Ax(t) + B\hat{u}(t), \quad t \geqslant L. \tag{4}$$

Through this transformation, Eqs. (3) and (4) are now both delay free and the LQR result for delay-free process can then be applied. It is well known (Lewis & Syrmos, 1995) that the LQR solution to process (4) is

$$\hat{u}(t) = -R^{-1}B^{\mathrm{T}}Px(t), \quad t \geqslant L, \tag{5}$$

where P is the positive definite solution of the Riccati equation:

$$A^{\mathrm{T}}P + PA - PBR^{-1}B^{\mathrm{T}}P + Q = 0. \tag{6}$$

Converting $\hat{u}(t)$ in Eq. (5) back to $u(t)$, we obtain the LQR solution to the original process (1) with the index (2) as

$$u(t) = \hat{u}(t+L) = -R^{-1}B^{\mathrm{T}}Px(t+L), \quad t \geqslant 0. \tag{7}$$

One sees from Eq. (7) that though the control law $\hat{u}(t)$ given in Eq. (5) is in time horizon of $t \geqslant L$, the recovered $u(t)$ actually gives the control signal for process (1) in the whole time horizon of $t \geqslant 0$. $x(t+L)$ is not directly available at time t. By Eqs. (3)–(5), however, it can be expressed by the transmission of $x(t)$ as

$$x(t+L) = \mathrm{e}^{(A-BR^{-1}B^{\mathrm{T}}P)t}x(L) = \mathrm{e}^{(A-BR^{-1}B^{\mathrm{T}}P)t}\mathrm{e}^{A(L-t)}x(t), \tag{8}$$

when $0 \leqslant t < L$ and

$$x(t+L) = \mathrm{e}^{(A-BR^{-1}B^{\mathrm{T}}P)t}x(L) = \mathrm{e}^{(A-BR^{-1}B^{\mathrm{T}}P)L}x(t), \tag{9}$$

when $t \geqslant L$. If we factorize the matrix Q as $Q = H^{\mathrm{T}}H$, the LQR solution to Eqs. (1) and (2) can thus be summarized (Marchall, 1979) in the following theorem.

Theorem 1. *For the linear process* (1) *with time delay, if* (A, B) *is controllable and* (H, A) *is observable, then the optimal control minimizing the criterion function* (2) *is given by*

$$u(t) = -R^{-1}B^{\mathrm{T}}P\mathrm{e}^{(A-BR^{-1}B^{\mathrm{T}}P)t}\mathrm{e}^{A(L-t)}x(t), \quad 0 \leqslant t < L \tag{10}$$

and

$$u(t) = -R^{-1}B^{\mathrm{T}}P\mathrm{e}^{(A-BR^{-1}B^{\mathrm{T}}P)L}x(t), \quad t \geqslant L, \tag{11}$$

where P is the positive definite solution to Eq. (6). The resultant system is also stable.

One may see from Eq. (7) that the current control $u(t)$ is actually a feedback of the future state at time of $(t+L)$. It implies that the controller has the prediction capability and thus may improve the closed-loop performance compared with traditional LQR or PID design. It is also noticed that during the starting period of time $t < L$, the control law (10) is time varying and generates a relatively large gain required to speed up the response. When $t = L$, Eq. (10) coincides with Eq. (11) and thus the control law is continuous. After that the feedback gain becomes constant, as seen in Eq. (11).

The major criticisms on LQR design, especially from the process control community, are that all the state variables are usually not measurable and the selection of the weighting matrices Q and R is not clear in order to meet the closed-loop performance specifications, say, overshoot and setting time. The goal of this paper is to develop an optimal PI/PID tuning algorithm via the above-outlined LQR solution for most typical industrial processes such that these problems can be solved and LQR design becomes truly useful for practical applications in process control.

3. PI tuning for first-order modeling

In process industry, a large class of processes has monotonic input–output transients whose transfer functions can be approximated (Luyben, 1990) by a first-order plus time-delay (FOPTD) one:

$$G(s) = \frac{b}{s+a}\,\mathrm{e}^{-Ls}. \tag{12}$$

It should be noted that Eq. (12) is not the process itself but a model of it and is used only for the purpose of controller design. The controller, once designed, should be applied to the process but not the model. A PI controller

$$u(t) = K_p\left(e(t) + \frac{1}{T_i}\int e(t)\,\mathrm{d}t\right) = K_p e(t) + K_i\int e(t)\,\mathrm{d}t \tag{13}$$

is adequate for such a kind of processes (Astrom & Hagglund, 1988). In this section, we will derive an optimal PI tuning algorithm via the LQR approach of the last section and the closed formulas for selecting Q and R in terms of the closed-loop specifications.

Consider a unity output feedback system shown in Fig. 1. In the case of feedback design, the external set-point does not affect the result and we put $r = 0$. It then follows from Fig. 1 that $(s+a)e = -b\mathrm{e}^{-Ls}u$, which is equivalent to the time-domain equation

$$\dot{e} = -ae - bu(t-L). \tag{14a}$$

We have the identity

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t e(t)\,\mathrm{d}t = e. \tag{14b}$$

Let $x_1 = \int_0^t e(t)\,\mathrm{d}t$ and $x_2 = e$ such that $x = [x_1 \quad x_2]^{\mathrm{T}}$. Then Eq. (14a) and (14b) can be written in the following equivalent form:

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & -a \end{bmatrix} x + \begin{bmatrix} 0 \\ -b \end{bmatrix} u(t-L). \tag{15}$$

It should be emphasized that both the variables are available (see Fig. 1) and the state feedback of Kx is simply $(K_i\int_0^t e\,\mathrm{d}t + K_p e)$, i.e., PI control. As a result, the state feedback gain to be derived by LQR will give us the required PI parameters.

In order to find the explicit expressions for K_i and K_p for ease of use, comparing Eq. (15) with Eq. (1) yields

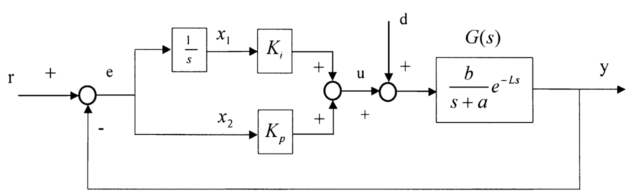

Fig. 1. Feedback control system.

$A = \left[\begin{smallmatrix} 0 & 1 \\ 0 & -a \end{smallmatrix}\right]$ and $B = \left[\begin{smallmatrix} 0 \\ -b \end{smallmatrix}\right]$. Let $Q = \left[\begin{smallmatrix} q_1 & 0 \\ 0 & q_2 \end{smallmatrix}\right]$. Substituting $P = \left[\begin{smallmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{smallmatrix}\right]$ into Riccati equation (6) yields

$$\begin{bmatrix} 0 & 0 \\ 1 & -a \end{bmatrix}\begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} + \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & -a \end{bmatrix} - \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix}\begin{bmatrix} 0 \\ -b \end{bmatrix} R^{-1}[0 \quad -b]\begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} + \begin{bmatrix} q_1 & 0 \\ 0 & q_2 \end{bmatrix} = 0. \tag{16}$$

Its positive definite analytical solution is

$$\begin{aligned} p_{12} &= \sqrt{q_1 R}/b, \\ p_{22} &= (-Ra + \sqrt{R^2a^2 + Rb^2(2p_{12} + q_2)})/b^2, \\ p_{11} &= ap_{12} + R^{-1}b^2 p_{12}p_{22}. \end{aligned} \tag{17}$$

Let

$$F = R^{-1}B^{\mathrm{T}}P = R^{-1}[0 \quad -b]\begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} = -R^{-1}b[p_{12} \quad p_{22}] \tag{18}$$

and

$$A_c = A - BF = \begin{bmatrix} 0 & 1 \\ 0 & -a \end{bmatrix} + \begin{bmatrix} 0 \\ -b \end{bmatrix} R^{-1}b[p_{12} \quad p_{22}] = \begin{bmatrix} 0 & 1 \\ -R^{-1}b^2p_{12} & -\sqrt{a^2 + R^{-1}b^2(2p_{12}+q_2)} \end{bmatrix}. \tag{19}$$

The optimal controller in Eq. (10) and (11) then reduces to

$$u(t) = \begin{cases} -Fe^{A_c t}e^{A(L-t)}x(t), & 0 \leqslant t < L, \\ -Fe^{A_c L}x(t), & t \geqslant L. \end{cases} \tag{20}$$

Remark 1. Note that given the process's A and B the optimal controller in Eq. (20) depends only on the gain F in Eq. (18), or only on p_{12} and p_{22} in the solution (17) of Riccati equation. Now, if the matrix Q in Eq. (16) is replaced by a general form of $Q = \left[\begin{smallmatrix} q_1 & q_{12} \\ q_{12} & q_2 \end{smallmatrix}\right]$, then its positive definite analytical solution is $p_{12} = (1/b)\sqrt{q_1 R}$, $p_{22} = [-Ra + \sqrt{R^2a^2 + Rb^2(2p_{12}+q_2)}]/b^2$ and $p_{11} = ap_{12} + R^{-1}b^2p_{12}p_{22} - q_{12}$. Note that p_{12} and p_{22} remain the same as those in Eq. (17) though the Q matrix is non-diagonal. This shows that choosing Q with a diagonal form will not lose the generality in the PI optimal controller design via the LQR approach for the case in Fig. 1.

To obtain the feedback gains in Eq. (20) explicitly, one needs to calculate $\exp(A_c t)$ and $\exp(A(L-t))$. It follows from the Laplace inverse transformation that

$$\exp(A(L-t)) = \ell^{-1}(sI-A)^{-1}|_{(L-t)} = \begin{bmatrix} 1 & (1-\exp(-a(L-t)))/a \\ 0 & \exp(-a(L-t)) \end{bmatrix}. \tag{21}$$

As for $\exp(A_c t)$, let $\hat{a}_2 = R^{-1}b^2 p_{12}$, $\hat{a}_1 = \sqrt{a^2 + R^{-1}b^2(2p_{12}+q_2)}$, α_1 and α_2 be the roots of the equation $s^2 + \hat{a}_1 s + \hat{a}_2 = 0$, i.e. $\alpha_1 = (-\hat{a}_1 + \sqrt{\hat{a}_1^2 - 4\hat{a}_2})/2$ and $\alpha_2 = (-\hat{a}_1 - \sqrt{\hat{a}_1^2 - 4\hat{a}_2})/2$. Then we have

$$\exp(A_c t) = \ell^{-1}(sI - A_c)^{-1} = \begin{bmatrix} f_{11}(t) & f_{12}(t) \\ f_{21}(t) & f_{22}(t) \end{bmatrix}, \tag{22}$$

where

$$f_{11}(t) = \frac{1}{\alpha_1 - \alpha_2}[(\alpha_1 + \hat{a}_1)e^{\alpha_1 t} - (\alpha_2 + \hat{a}_1)e^{\alpha_2 t}],$$

$$f_{12}(t) = \frac{1}{\alpha_1 - \alpha_2}[e^{\alpha_1 t} - e^{\alpha_2 t}],$$

$$f_{21}(t) = \frac{-\hat{a}_2}{\alpha_1 - \alpha_2}[e^{\alpha_1 t} - e^{\alpha_2 t}] \quad \text{and}$$

$$f_{22}(t) = \frac{1}{\alpha_1 - \alpha_2}[\alpha_1 e^{\alpha_1 t} - \alpha_2 e^{\alpha_2 t}].$$

Recall that $u = Kx = [K_i, K_p][\int_0^t e\,\mathrm{d}t, e]^{\mathrm{T}}$. Substituting Eqs. (18), (19), (21) and (22) into Eq. (20) gives us the explicit expressions for the PI parameters.

Theorem 2. *The LQR optimal control for process* (12) *with state equation (15) is given in the form of a PI controller* (13), *where for* $0 \leqslant t < L$,

$$K_i(t) = R^{-1}b[p_{12} f_{11}(t) + p_{22} f_{21}(t)], \tag{23a}$$

$$K_p(t) = R^{-1}b\left\{\frac{1}{a}p_{12} f_{11}(t) + \frac{1}{a}p_{22} f_{21}(t) + \left[p_{12} f_{12}(t) - \frac{1}{a}p_{12} f_{11}(t) + p_{22} f_{22}(t) - \frac{1}{a}p_{22} f_{21}(t)\right]e^{-a(L-t)}\right\} \tag{23b}$$

and for $t \geqslant L$,

$$K_i(t) = R^{-1}b[p_{12} f_{11}(L) + p_{22} f_{21}(L)], \tag{24a}$$

$$K_p(t) = R^{-1}b[p_{12} f_{12}(L) + p_{22} f_{22}(L)], \tag{24b}$$

where constants p_{12} *and* p_{22} *are given in Eq. (17)*, $f_{ij}(t)$, $i = 1, 2$; $j = 1, 2$, *are given in Eq. (22)*, q_1, q_2 *and* R *are tuning parameters*.

In an ordinary LQR design, the selection of Q and R matrix is quite technical and affects the system performance a lot. In order to overcome this difficulty, we now derive an direct relationship between q_1 and q_2, and the damping ratio ξ and natural frequency ω_n of the closed-loop system.

Theorem 3. *When* $t \geqslant L$, *the damping ratio* ξ *and natural frequency* ω_n *of the LQR optimal closed-loop system in Eqs. (13) and (15) is*

$$\omega_n = \sqrt{R^{-1}b\sqrt{q_1 R}},$$

$$\xi = \frac{\sqrt{a^2 + R^{-1}b(2\sqrt{q_1 R} + q_2 b)}}{2\sqrt{R^{-1}b\sqrt{q_1 R}}} \tag{25}$$

or equivalently, in order to have the desired ξ *and* ω_n, q_1 *and* q_2 *should be chosen as*

$$q_1 = \frac{\omega_n^4 R}{b^2},$$

$$q_2 = \frac{[(4\xi^2 - 2)\omega_n^2 - a^2]R}{b^2}. \tag{26}$$

Proof. When $t \geqslant L$, the closed-loop system becomes

$$\dot{x} = A_c x = \begin{bmatrix} 0 & 1 \\ -R^{-1}b\sqrt{q_1 r} & -\sqrt{a^2 + R^{-1}b(2\sqrt{q_1 r} + q_2 b)} \end{bmatrix},$$

whose characteristic equation is

$$\Delta = s(s + \sqrt{a^2 + R^{-1}b(2\sqrt{q_1 r} + q_2 b)}) + R^{-1}b\sqrt{q_1 R}.$$

It thus has

$$\omega_n^2 = R^{-1}b\sqrt{q_1 R},$$

$$2\xi\omega_n = \sqrt{a^2 + R^{-1}b(2\sqrt{q_1 R} + q_2 b)}.$$

The theorem is then obvious. □

Remark 2. For the system (15) with $Q = \mathrm{diag}\{q_1, q_2\}$ with q_1 and q_2 chosen according to Eq. (26), the performance index (2) becomes

$$J = R\left[\int_0^\infty \left[\frac{\omega_n^4}{b^2}\left(\int_0^t e(t)\,\mathrm{d}t\right)^2 + \frac{(4\xi^2 - 2)\omega_n^2 - a^2}{b^2}e(t)^2 + u(t)^2\right]\mathrm{d}t\right],$$

i.e., J is proportional to R. This implies that R makes no sense in the design of controller gain F in Eq. (18) and thus we can always choose $R = 1$ when Theorem 3 is applied.

In view of the above development, an optimal PI tuning algorithm for process (12) can be summarized as follows for ease of reference.

An optimal PI tuning algorithm

Initialization: Obtain a, b, L and set $R = 1$.

Step 1. Choose the closed-loop ω_n and ξ.

Step 2. Calculate q_1 and q_2 from (26).

Step 3. Calculate p_{12} and p_{22} from (17), A_c from (19) and $\exp(A_c t)$ from (22).

Step 4. Calculate the PI parameters from (23) and (24).

Remark 3. In the proposed algorithm, ξ and ω_n are only user-specified parameters. To our experience, choosing $\xi \in [0.7, 1.0]$ and $\omega_n L \in [1.0, 1.5]$ would give a satisfactory result. Normally, we can use defaults $\xi = 0.71$ and $\omega_n L = 1.3$. For a better performance, a finer tuning procedure may be employed.

4. Simulation studies

The PI tuning algorithm proposed in the last section will be demonstrated for different processes and shown with thicker solid lines in the simulations (Figs. 2–5) below. For comparison, the PI controller tuning by Ho's gain-phase margin method (GPM) (Ho et al., 1995a) and Luyben's hybrid approach (Luyben, 1998) are employed. Ho et al. (1995b) compares various well-known PI tuning algorithms and shows that the GPM method has a better performance over most of the other PI tuning ones. Thus, it represents one of the best available PI tuning methods in the literature. Default gain margin of 3 and phase margin of 45°, as suggested by Ho, are chosen for Ho's algorithm throughout the simulation examples and the corresponding closed-loop responses are shown by thin solid lines. The simulation with the Luyben's method is also presented since it is one of the latest development in the domain and its performance is plotted with dashed lines. To tune a PI controller, the above three methods have to be supplied with a model. A model for the process may be obtained by process-model matching at two frequencies (Luyben, 1990) or by least-squares fitting between process and model frequency responses (Wang, Hang & Bi, 1997). The latter is used for our simulations. For a fair comparison, the same processes with the same identified models will be used in the simulations. Responses to a setpoint change of $r = 1$ and a disturbance of $d = 0.2$ are shown.

Example 1. Consider a high vacuum distillation column. The transfer function between the viscosity and the reflux flow is given by

$$G_p(s) = \frac{0.57}{(1 + 8.60s)^2} e^{-18.70s}$$

and the model is identified as

$$G(s) = \frac{0.57}{1 + 12.72s} e^{-23.2s}.$$

With $\omega_n L = 1.3$ and $\xi = 0.71$, the PI controller's parameters are obtained as

$$K_i(t) = [0.0701 \cos(0.0395t) + 0.0690 \sin(0.0395t)] e^{-0.0398t},$$

$$K_p(t) = [0.1417 \sin(0.0395t) - 0.1404 \cos(0.0395t)] e^{0.0388t}$$

$$+ [0.8913 \cos(0.0395t) + 0.8771 \sin(0.0395t)] e^{-0.0398t}$$

for $0 \leqslant t < 23.2$, and $K_i = 0.0387$, $K_p = 0.5581$ for $t \geqslant 23.2$. The control signal and the closed-loop performance are shown in Fig. 2A. The results show that the LQR method gives a perfect response and significantly outperforms that of the other two methods.

To see what causes different shapes of response, PI parameters for three tuning approaches are shown in Fig. 2B. The hybrid one has very big K_p and very small K_i, leading to the response with a short rise time but long settling time and significant undershoot and oscillations. The GPM gives rise to a smaller K_p and bigger K_i relatively to other two methods, causing the response with a long rise time and obvious overshoot. On the other hand, the proposed LQR method correctly produces quite big controller gains (both K_p and K_i) at the initial stage, which are necessary to speed up the response, and afterwards it quickly reduces the gains to a reasonable level to avoid overshoot and have a short settling time. The variable gains thus make a perfect response, which other methods could never achieve. To further convince the necessity of using the time-varying PID and impossibility of any fixed parameter PID for a perfect response, in addition to GPM tuning with defaults values, we re-tune PID using the GPM with other margins to have a similar rise time (curve 1) and a similar overshoot/undershoot (curve 3) to our method, respectively. The three cases as well as our tuning case are all shown in Fig. 2C and the corresponding gains in Fig. 2D. It is absolutely clear that the tuning of a fixed PID faces unavoidable trade-off between rise time and settling time/overshoot. Thus a fixed PID cannot achieve all the best properties simultaneously, and time-varying PID is necessary for such a purpose and is actually possible as proven by the proposed method, where time varying part of the gains enables a fast response without causing overshoot.

Example 2. Consider the high-order process

$$G_p(s) = \frac{1}{(s + 1)^n},$$

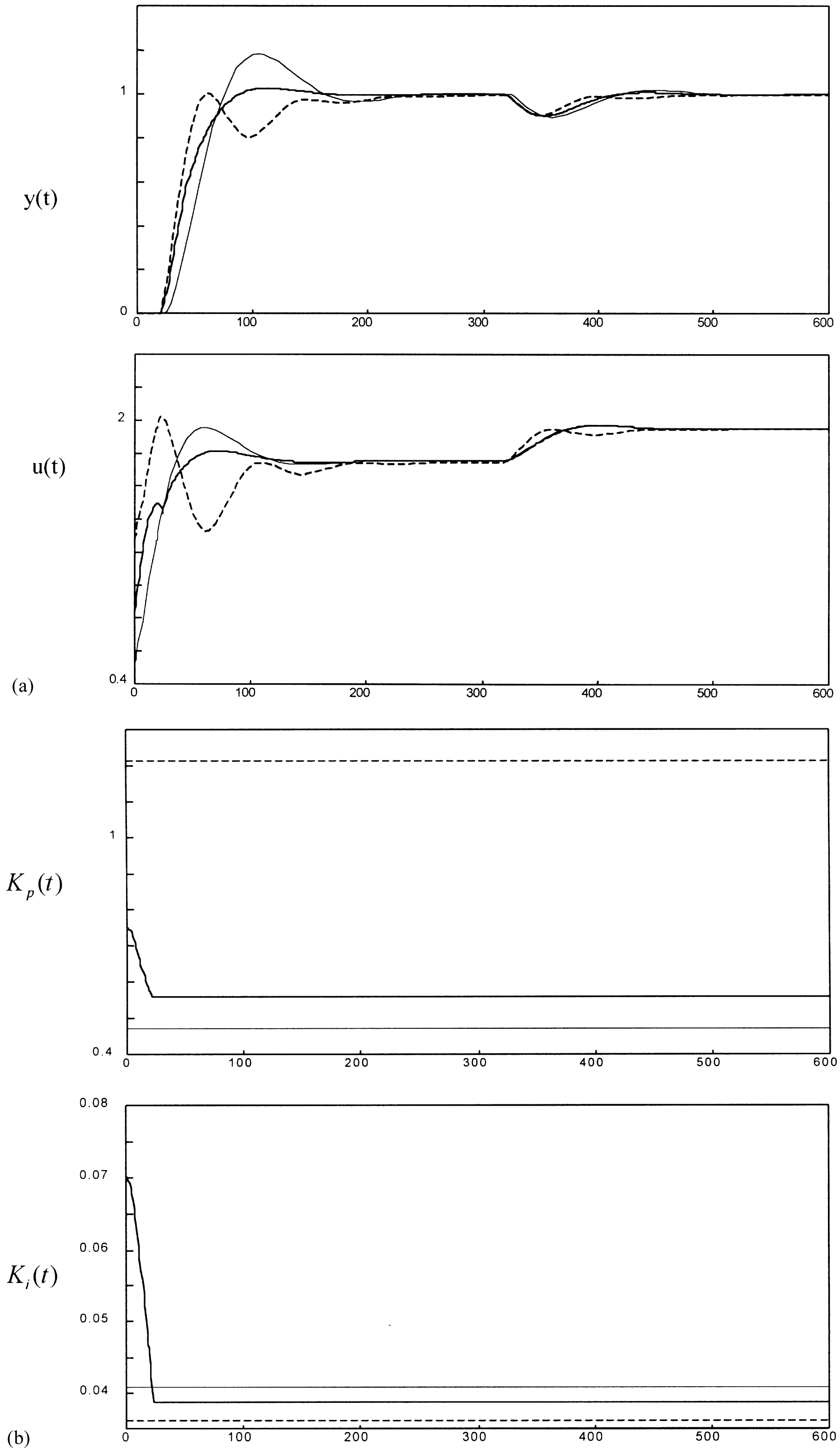

Fig. 2. (A) Closed-loop Performance for $[0.57/(1+8.60s)^2]e^{-18.70s}$. (B) PI parameters for plant $[0.57/(1+8.60s)^2]e^{-18.70s}$. (C) Compare with fixed PI tuning approach (GPM method). (D) PI parameters for different GPM tuning.

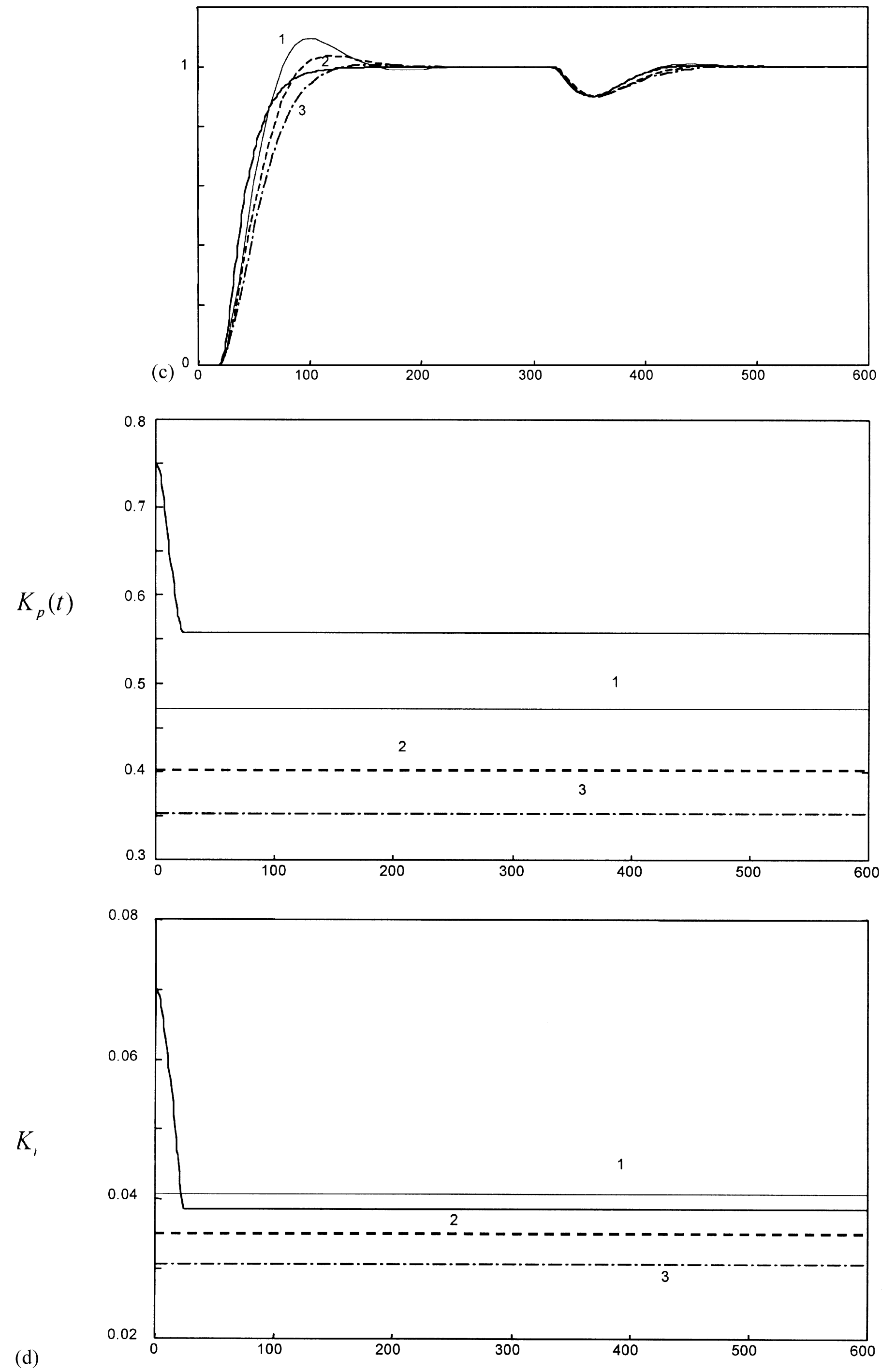

Fig. 2. Continued.

with $n = 10$ and 20, respectively. The resultant models are

$$\frac{1}{1 + 2.72s} e^{-7.69s}, \quad n = 10,$$

$$\frac{1}{1 + 4.95s} e^{-15.67s}, \quad n = 20.$$

The closed-loop responses in Fig. 3 show that the proposed method has a much better performance than that of the Ho's method.

Example 3. The algorithm is also applied to the non-minimum-phase process

$$G_p(s) = \frac{1 - \alpha s}{(1 + s)^3},$$

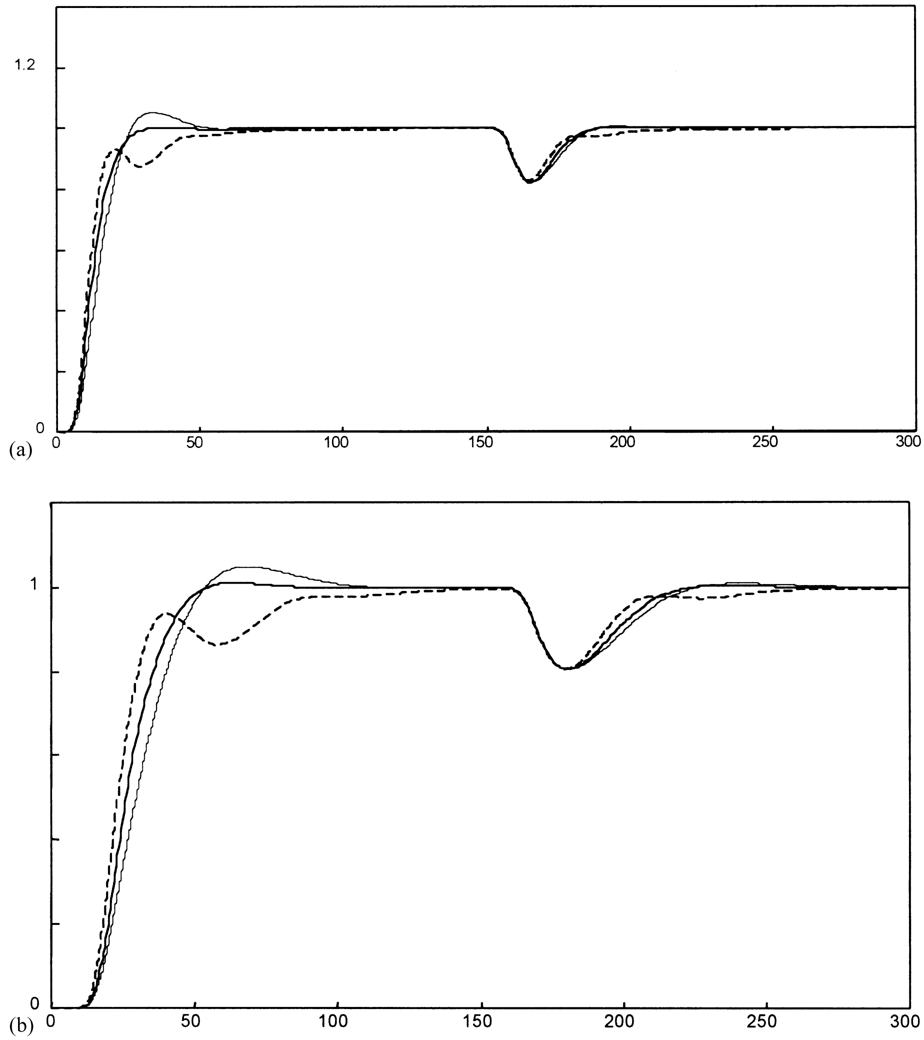

Fig. 3. (A) Closed-loop Performance for $1/(s+1)^n$ with $n=10$. (B) Closed-loop Performance for $1/(s+1)^n$ with $n=20$.

with $\alpha = 1$ and $\alpha = 1.5$. The closed-loop performance shown in Fig. 4 exhibits a great improvement with the proposed method.

The simulations from the above examples show that the proposed LQR-based PI tuning algorithm gives a much better closed-loop performance over some well-known PI tuning methods. One also sees that the control signal given by the LQR method is larger than that of GPM method when $t < L$, and it leads to a faster setpoint response. But thereafter the gains decrease and the overall control signal amplitude is not larger than that for ordinary PI tuning. For these three examples, the actual processes, their models, output performance specifications and PI parameters are listed in Table 1 for ease of reference.

5. Extension to second-order modeling

PI control is sometimes inadequate when the process dynamics is not essentially first order (Astrom & Hagglund, 1988). In this section, the PID tuning formula will be derived for the second-order plus time-delay model instead of the first-order plus time-delay model. Consider a second-order process model given by

$$G(s) = \frac{b}{(s+a)(s+a_1)} e^{-Ls}, \quad a_1 \geqslant a. \tag{27}$$

A PID controller is written in the from of

$$G_c(s) = \left(K_p + K_i \frac{1}{s}\right)(s + K_d) \tag{28}$$

Then, from Eqs. (27) and (28), the open-loop transfer function is

$$G_c(s)G(s) = \left(K_p + K_i \frac{1}{s}\right) \frac{b(K_d + s)}{(s+a)(s+a_1)} e^{-Ls}. \tag{29}$$

For ease of control design, one may choose, like Ho et al. (1995b), $K_d = a_1$ to cancel the larger process pole.

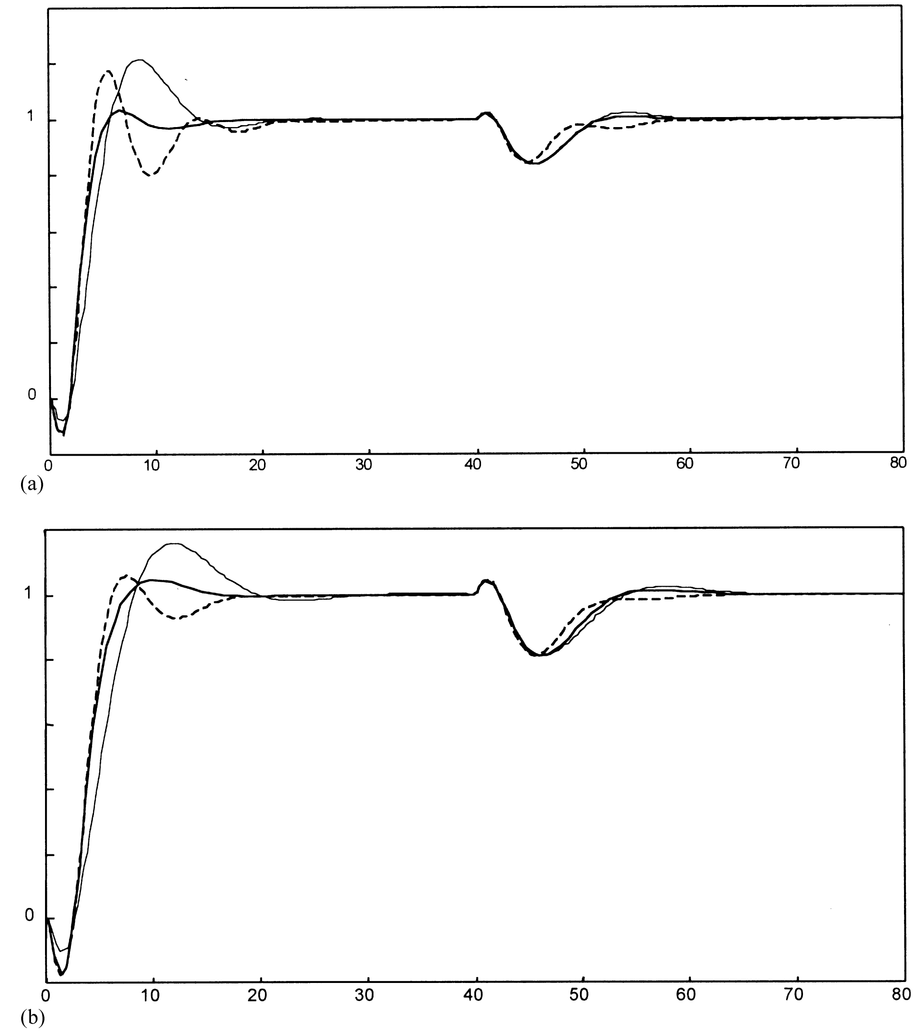

Fig. 4. (A) Closed-loop Performance for $(1-\alpha s)/(1+s)^3$ with $\alpha = 1$. (B) Closed-loop Performance for $(1-\alpha s)/(1+s)^3$ with $\alpha = 1.5$.

Eq. (29) is then reduced to

$$G_c(s)G(s) = \left(K_p + K_i\frac{1}{s}\right)\frac{b}{(s+a)}e^{-Ls}. \tag{30}$$

Note that Eq. (30) now gives the same open-loop transfer function as with the FOPTD model $G(s)$ given in Eq. (12) and a PI controller in Eq. (13). Therefore, a PID controller can be simply tuned with $K_d = a_1$ and the PI parameters given as in the last section.

Similarly to the FOPTD model, a second-order plus time-delay model in Eq. (27) can be identified by various identification methods (Luyben, 1990). In Example 4, the least-squares fitting method is employed to identify the model and then the proposed PID tuning method is applied. Ho's tuning formula for PID (Ho et al., 1995b) is adopted again for comparison.

Example 4. Consider a non-minimum phase process

$$G_p(s) = \frac{1-s}{(1+s)^2(2+s)}$$

and its model is obtained as

$$G(s) = \frac{e^{-1.64s}}{(s+1)(s+2)}.$$

Choose $\omega_n L = 1.3$ and $\xi = 0.8$, the PID parameters are obtained as $K_p = 0.6138$, $K_i = 0.5561$ and $K_d = 1$ for $t \geqslant 1.64$. The simulation result in Fig. 5 shows a great improvement of our method over Hos (Ho et al., 1995b) which has a gain margin of 3 and phase margin of 45°.

6. Robustness analysis

One of the most attractive properties of LQR design for delay-free processes is the robustness of its closed-loop system, which is usually wanted in practical applications. If the process is of single-input and single-output, the resultant LQR system has at least the phase margin of 60° and gain margin of infinity. Unfortunately, it is found that this property cannot be carried over to the

Table 1
PI tuning and performance

Process	Model	Method	$\omega_n L(A_m)$	$\xi(\Phi_m)$	$K_p/K_p(\infty)$	$K_i/K_i(\infty)$	Rise time	Settling time	Overshoot/undershoot
$\frac{0.57e^{-18.70s}}{(1+8.60s)^2}$	$\frac{0.57}{1+12.72s}e^{-23.2s}$	LQR	1.3	0.71	0.5581	0.0387	44.7	76	0.0/0.0
		GPM	3	45	0.4719	0.0408	39.5	129	0.09/0.009
		Hybrid	×	×	1.2464	0.0362	25	132	0.0/0.2
$\frac{1}{(1+s)^{10}}$	$\frac{1}{1+2.72s}e^{-7.69s}$	LQR	1.4	0.71	0.2487	0.0701	13.3	23.5	0.002/0.0
		GPM	3	45	0.1735	0.0680	13.9	36.7	0.054/0.002
		Hybrid	×	×	0.0472	0.0658	10.3	40	0.0/0.125
$\frac{1}{(1+s)^{20}}$	$\frac{1}{1+4.95s}e^{-15.67s}$	LQR	1.4	0.71	0.2335	0.0349	22.6	22	0.01/0.0
		GPM	3	45	0.1550	0.0331	26	48	0.049/0.0
		Hybrid	×	×	0.4523	0.0327	19	79	0.0/0.135
$\frac{1-s}{(1+s)^3}$	$\frac{1}{1+1.61s}e^{-2.25s}$	LQR	1.2	0.71	0.3806	0.2240	2.5	5	0.03/0.03
		GPM	3	45	0.3511	0.2468	3.0	12.6	0.21/0.028
		Hybrid	×	×	0.8758	0.2054	1.8	12.3	0.175/0.2
$\frac{1-1.5s}{(1+s)^3}$	$\frac{1}{1+1.01s}e^{-2.89s}$	LQR	1.5	0.71	0.2574	0.1850	3.4	6.9	0.045/0.024
		GPM	3	45	0.1715	0.1807	4.3	17	0.158/0.02
		Hybrid	×	×	0.4922	0.1754	2.7	8	0.058/0.075

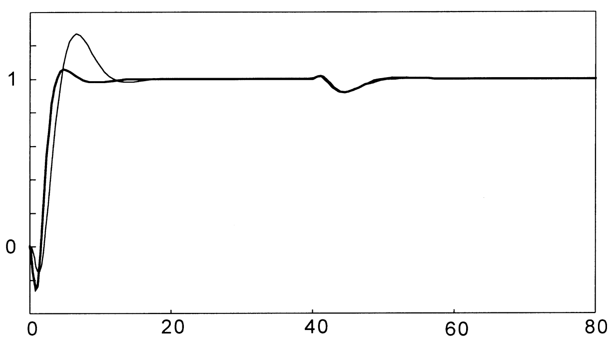

Fig. 5. Closed-loop Performance for $(1-s)/(1+s)^2(2+s)$.

time-delay case in general. Its extension is only possible for special systems.

Let us consider the stability of the controlled system in Eqs. (1) and (11) with the real parameters perturbed to A_r, B_r and L_r. Without the loss of generality, the robustness issue is considered when $t \geqslant \max\{L, L_r\}$, because during a finite time interval, the system cannot go to infinity. If the control law of Eq. (11) is applied to the process (1), the resultant closed loop is

$$\dot{x}_r = A_r x_r - B_r R^{-1} B^T P \exp(A_c L) x_r(t - L_r), \tag{31}$$

where $A_c = A - BR^{-1}B^T P$. Let matrix $\hat{A}_c$ be the solution of equation

$$\hat{A}_c = A_r - B_r R^{-1} B^T P \exp(A_c L)\exp(-\hat{A}_c L_r), \tag{32}$$

then we have the following theorem.

Theorem 4. *The perturbed system with real process parameters A_r, B_r and L_r remains stable if all eigenvalues of $\hat{A}_c$ given by Eq. (32) lie in the open left half of the complex plane.*

Proof. Substituting Eq. (32) into Eq. (31) yields

$$\begin{aligned}\dot{x}_r(t) &= [\hat{A}_c + B_r R^{-1} B^T P \exp(A_c L)\exp(-\hat{A}_c L_r)]x_r(t)\\ &\quad - B_r R^{-1} B^T P \exp(A_c L) x_r(t - L_r).\end{aligned} \tag{33}$$

Consider the dynamic equation

$$\dot{x}(t) = \hat{A}_c x(t). \tag{34}$$

Its state transition satisfies

$$x(t - L_r) = \exp(-\hat{A}_c L_r) x(t).$$

Observe that the two differential equations (33) and (34) coincide. Thus Eq. (33) is stable if Eq. (34) is stable, i.e. all the eigenvalues of $\hat{A}_c$ given by Eq. (34) lie in the open left half of the complex plane. The proof is completed. □

Now, we consider Eq. (32) in the special case of the PI controller (13) for process (12) with real parameters $A_r = \begin{bmatrix} 0 & 1 \\ 0 & -a_r \end{bmatrix}$, $B_r = \begin{bmatrix} 0 \\ -b_r \end{bmatrix}$ and L_r. Let

$$\exp(A_c L)\exp(-\hat{A}_c L_r) = I + \Delta = \begin{bmatrix} 1 + \Delta_{11} & \Delta_{12} \\ \Delta_{21} & 1 + \Delta_{22} \end{bmatrix}. \tag{35}$$

Note that in the case of no modeling error in process (12), i.e. $A_r = A$, $B_r = B$ and $L = L_r$, $\hat{A}_c = A_c$ will be the solution of Eq. (32), or $\exp(A_c L)\exp(-\hat{A}_c L_r) = I$ and

$\Delta_{ij}=0$, $i,j=1,2$. In general, however, the solution of Eq. (32) is continuous with respect to A_r, B_r and L_r. In other words, for any $\varepsilon>0$, there exists a $\delta>0$ such that if $\|A_r-A\|<\delta$, $\|B_r-B\|<\delta$ and $|L_r-L|<\delta$, then $|\Delta_{ij}|<\varepsilon$, $i,j=1,2$. The following proposition further shows that when the process perturbation is small enough, the control system will remain stable.

Proposition 1. *If the process parameter perturbations are small enough, the control system in Eqs. (12) and (13) is robustly stable.*

Proof. Substituting A_r, B_r, L_r and Eq. (35) into Eq. (32) gives

$$\hat{A}_c=\begin{bmatrix}0 & 1\\ 0 & -a_r\end{bmatrix}-\begin{bmatrix}0\\ R^{-1}b_rb\end{bmatrix}[p_{12}\quad p_{22}](I+\Delta)$$
$$=\begin{bmatrix}0 & 1\\ -R^{-1}b_rbp_{12}-R^{-1}b_rb(p_{12}\Delta_{11}+p_{22}\Delta_{21}) & -a_r-R^{-1}b_rb(p_{22}+p_{12}\Delta_{12}+p_{22}\Delta_{22})\end{bmatrix}.$$

The eigenvalues of $\hat{A}_c$ lie in the open left half of the complex plane if

$$-R^{-1}b_rbp_{12}-R^{-1}b_rb(p_{12}\Delta_{11}+p_{22}\Delta_{21})<0,$$
$$-a_r-R^{-1}b_rb(p_{22}+p_{12}\Delta_{12}+p_{22}\Delta_{22})<0$$

or

$$\Delta_{11}+\frac{p_{22}}{p_{12}}\Delta_{21}>-1,$$
$$\Delta_{22}+\frac{p_{12}}{p_{22}}\Delta_{12}\geqslant-1. \tag{36}$$

Note from Eq. (21) that p_{12} and p_{22} are positive real numbers, and Eq. (36) will hold true if we choose an ε such that $\varepsilon<\min\{0.5, p_{12}/2p_{22}, p_{22}/2p_{12}\}$. Thus, for such an ϵ and $|\Delta_{ij}|<\varepsilon$, $i,j=1,2$, there exists a corresponding $\delta>0$ such that if the parameter perturbations in the process satisfy $\|A_r-A\|<\delta$, $\|B_r-B\|<\delta$ and $|L_r-L|<\delta$, the resultant closed-loop system remains stable. The proposition is proved. □

7. Conclusions

Time delay is a very common phenomenon in process industry. In this paper, an LQR solution has been used to develop an optimal tuning algorithm for processes with time delay. The algorithm can produce optimal PID settings and generate the expected closed-loop performance from the user's specifications on damping ratio and natural frequency. It has been seen from examples of various dynamics given here that the proposed tuning significantly outperforms some of the best existing method. This is largely due to the prediction capability of our LQR controller and the established relationship between the weightings and closed-loop performance.

References

Astrom, K. J., & Hagglund, T. (1988). Automatic tuning of PID controllers: Instrument Society of America.

Hang, C. C., Astrom, K. J., & Ho, W. K. (1991). Refinements of the Ziegler–Nichols tuning formula. *IEE Proceedings D*, *138*(2), 111–118.

Ho, W. K., Hang, C. C., & Cao, L. S. (1995a). Tuning of PID controllers based on gain and phase margin specifications. *Automatica*, *31*(3), 497–502.

Ho, W. K., Hang, C. C., & Zhou, J. (1995b). Performance and gain and phase margins of well-known pi tuning formulas. *IEEE Transactions on Control Systems Technology*, *3*, 245–248.

Huang, H. P., Chen, C. L., Lai, C. W., & Wang, G. B. (1996). Autotuning for model based PID controllers. *AIChE Journal*, *42*(9), 2687–2691.

Lewis, F. L., & Syrmos, V. L. (1995). *Optimal Control*. New York: Wiley.

Luyben, W. L. (1998). Hybrid method for tuning PI controllers for lag/deadtime processes. *AIChE meeting*.

Luyben, W. L. (1990). *Process modeling, simulation and control for chemical engineers* (2nd ed.). New York: McGraw-Hill Publishing Company.

Marshall, J. E. (1979). Control of time-delay systems. London: Peter Peregrinus LTD.

Morari, M., & Zafiriou, E. (1989). *Robust process control*. Englewood Cliffs, NJ: Prentice-Hall Inc..

Rotstein, G. E., & Lewin, D. R. (1991). Simple PI and PID tuning for open-loop unstable systems. *Industrial Engineering Chemistry Research*, *30*, 1864–1869.

Shinskey, F. G. (1988). *Process control system, application, design and tuning* (3rd ed.). New York: McGraw-Hill.

Wang, Q. G., Hang, C. C., & Bi, Q. (1997). A frequency domain controller design method. *Chemical Engineering Research and Design*, *75*, 64–72.

Zhuang, M., & Atherton, D. P. (1993). Automatic tuning of optimal PID controllers. *IEE Proceeding D*, *140*(3), 216–224.

Ziegler, J. G., & Nichols, N. B. (1942). Optimum settings for automatic controllers. *Transactions of ASME*, *65*, 433–444.