Master study
Systems and Control Engineering
Department of Technology
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SCEV3006 Advanced Control with Implementation

Exercise 4

Task 1

A mathematical model of two inverted pendulums which stands below each other on a chart is given by

$$\ddot{q}_1 = \frac{m_1 l_{G1} g \sin(q_1)}{J_{t1} + m_1 l_{G1}^2} - \frac{m_1 l_{G1} \cos(q_1)}{J_{t1} + m_1 l_{G1}^2} \dot{v} \tag{1}$$

$$\ddot{q}_2 = \frac{m_2 l_{G2} g \sin(q_2)}{J_{t2} + m_2 l_{G2}^2} - \frac{m_2 l_{G2} \cos(q_2)}{J_{t2} + m_2 l_{G2}^2} \dot{v} \tag{2}$$

where q_1 is the angle for pendulum number 1, i.e. the angle of pendulum 1 from the vertical line. q_2 is the angle for pendulum number 2 and v is the velocity of the chart.
The constants in the model is given by:

- $m_1 = 0.244$ is the mass of pendulum nr. 1 (inc. bar and load),
- $m_2 = 0.225$ is the mass of pendulum nr. 2 (inc. bar and load),
- $l_{G1} = 0.6369$ is the distance from origo to the center of gravity $G1$ for pendulum nr. 1.
- $l_{G2} = 0.3778$ is the distance from the holding on the chart to the point of gravity $G2$ for pendulum nr. 2.
- $J_{t1} = 0.0062$ $[kgm^2]$ is the moment of inertia for bar 1 with load about the center of gravity $G1$.
- $J_{t2} = 0.0012$ $[kgm^2]$ is the moment of inertia for bar 2 with load about the center of gravity $G2$.
- $g = 9.81$ is the acceleration of the gravity.

Part 1

We will in this part use a simplified model for the control actuator, i.e. the chart, such that $M\dot{v} = u$ where $M = 1$ is the mass of the chart and u is the manipulable force (control) on the chart.
Sketch a figure for the system.

a) Use the following state definitions

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} q_1 \\ \dot{q}_1 \\ q_2 \\ \dot{q}_2 \end{bmatrix} \tag{3}$$

and write up a state space model as a set of 1st order differential equations. Do not insert numerical values at this stage.

b) Use the state vector defined in step **a)** above and find a linearized state space model of the form

$$\dot{x} \;=\; Ax + Bu \tag{4}$$

c) Put numerical values into the linearized state space model and analyze the system for stability, controllability and observability. We are assuming that both $x_1 = q_1$ and $x_2 = q_2$ are measured (the angle velocities are not measured).

d) Compute the LQ optimal controller for the system and simulate the closed loop system with different initial values, i.e., $x_1(t = 0) = q_1(t = 0) \neq 0$ and $x_2(t = 0) = q_2(t = 0) \neq 0$. You can here assume that all states are measured and available for feedback control. Study both the linear and the non-linear system under LQ optimal control feedback.

Part 2

We will in this part of the exercise study two different alternative equations of motion for the chart.
Alternative 2. An alternative model for the control actuator is given by

$$M\dot{v} = -m_1 g \tan(q_1) - m_2 g \tan(q_2) + u \tag{5}$$

This model can be deduced from a simplified force balance on the chart in the horizonal plane. How will this model influence upon the results in Part 1 of the exercise?
Alternative 3. A third alternative equation of motion for the chart with the two inverted pendulums is given by

$$M\dot{v} = u - m_1 a_{x1} - m_2 a_{x2} - bv \tag{6}$$

where a_{x1} and a_{x2} is the acceleration components of the pendulums in the direction of movement (i.e. the x-axis), given by

$$a_{x1} = \dot{v} + l_{G1}\cos(q_1)\ddot{q}_1 - l_{G1}\sin(q_1)\dot{q}_1^2 \quad (7)$$
$$a_{x2} = \dot{v} + l_{G2}\cos(q_2)\ddot{q}_2 - l_{G2}\sin(q_2)\dot{q}_2^2 \quad (8)$$

How should the problem now be solved?

Task 2

Given a system

$$\dot{x} = Ax + Bu \quad (9)$$

$$(10)$$

where

$$A = \begin{bmatrix} -6 & 5 \\ 5 & -6 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \quad (11)$$

We want to control the system with an LQ optimal controller which minimizes the criterion

$$J = \frac{1}{2}\int_0^\infty (x^T Q x + u^T P u) dt \quad (12)$$

where

$$Q = \begin{bmatrix} 135 & 60 \\ 60 & 135 \end{bmatrix}, \quad P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \quad (13)$$

a) Write down an expression for the optimal control. The solution is a function of the matrix R which is found as the positive definite solution to a Riccati equation, ore via the transition matrix (matrix exponential) of the Hamiltonian matrix F. Write down the equations for both alternatives...

b) Put into numerical values and find the LQ optimal controller. Use MATLAB for the computations, and perform both alternatives sketched in step 2a) above.

c) Compute the eigenvalues of the closed loop system. Can you say something about the eigenvalues of the closed loop system compared to the open loop system?