Master study
Systems and Control Engineering
Department of Technology
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IIAV3017 Advanced Control with Implementation

Exercise 2b

Task 1

Pole polynomial and system poles

The pole polynomial is given by the least common denominator for all sub determinants of all orders of the system transfer matrix. We note that the system has $r = 2$ inputs and $m = 3$ outputs. The system transfer matrix $H(s)$ has therefore the maximal rank $r_H = 2$. For this system we have both 1st order and 2nd order sub determinants. The sub determinants of 1st order is identic with the six elements in the transfer matrix. Furthermore, we have three 2nd order sub determinants. These are as follows

$$H^1_{1,2}(s) = \frac{2s^2 - s - 8 + s^2 + s - 4}{(s^2 + 3s + 2)^2} = \frac{3(s+2)(s-2)}{(s+2)^2(s+1)^2} = \frac{3(s-2)}{(s+2)(s+1)^2}, \quad (1)$$

$$H^2_{1,3}(s) = \frac{2s - 4 + s - 2}{(s^2 + 3s + 2)(s+1)} = \frac{3(s-2)}{(s+2)(s+1)^2}, \quad (2)$$

$$H^3_{2,3}(s) = \frac{(s^2 + s - 4)(2s - 4) - (s-2)(2s^2 - s - 8)}{(s^2 + 3s + 2)(s+1)} = \frac{3s(s-2)}{(s+2)(s+1)^2}. \quad (3)$$

Note that super script 1, 2 and 3 in the notations $H^1_{1,2}(s)$, $H^2_{1,3}(s)$ and $H^3_{2,3}(s)$ denotes sub determinant one, two and three. Sub script $(1, 2)$ means that sub determinant $H^1_{1,2}(s)$ is computed from the sub matrix formed by using row one and row two in the transfer matrix. Sub script $(1, 3)$ means that sub determinant $H^1_{1,3}(s)$ is computed from the sub matrix formed by using row one and row three in the transfer matrix. Sub script $(2, 3)$ means that sub determinant $H^1_{2,3}(s)$ is computed from the sub matrix formed by using row two and row three in the transfer matrix.
We construct a common denominator from the 1st and the 2nd order determinants. This gives the pole polynomial:

$$\pi(s) = (s+1)^2(s+2). \quad (4)$$

The system poles are then given by the solution of $\pi(s) = 0$, i.e., the system have three poles $s_1 = -1$, $s_2 = -1$ og $s_3 = -2$.

Zero polynomial and system zeroes

The zero polynomial is found as the largest common numerator (divisor) to all sub determinants of order equal to the natural rank, $r_H = \min(m, r)$, of the transfer matrix $H(s)$, in this case the sub determinants of 2nd order, supposed they are justified such that they have the pole polynomial as denominator.
As we see, the sub determinants $H^1_{1,2}(s)$, $H^2_{1,3}(s)$ and $H^3_{2,3}(s)$ have the pole polynomial $\pi(s)$ as denominator. The zero polynomial is therefore given by

$$\rho(s) = s - 2. \tag{5}$$

The system have one transmission zero at $s_0 = 2$ which is given by $\rho(s) = 0$. We note that system is a non-minimum-phase system.

Task 2

The determinant (of 2. order) to the transfer matrix is

$$\begin{aligned} \det(H(s)) &= \frac{-s^2(s+1)^2 + s^2(s+1)^4}{(s+1)^4(s+2)^4} = \frac{s^2(s+1)^2(-1+(s+1)^2)}{(s+1)^4(s+2)^4} && (6) \\ &= \frac{s^3(s+1)^2(s+2)}{(s+1)^4(s+2)^4} = \frac{s^3}{(s+1)^2(s+2)^3}. && (7) \end{aligned}$$

Pole polynomial and poles

The largest common divisor (for the numerators) for the underdeterminants of 1st order (the elements in $H(s)$) and the 2nd order determinants, gives the pole polynomial

$$\pi(s) = (s+1)^2(s+2)^3. \tag{8}$$

The system poles is given by $\pi(s) = 0$, i.e., two poles $s_{1,2} = -1$ and three poles $s_{3,4,5} = -2$.

The zero polynomial and the zeroes

The 2nd order determinant has the pole polynomial as denominator. Hence, the zero polynomial is given by

$$\rho(s) = s^3. \tag{9}$$

The system has three poles in origo, i.e. the poles is $s^0_{1,2,3} = 0$.

Task 3

We define the system matrix

$$S = \begin{bmatrix} A & B \\ D & E \end{bmatrix} = \begin{bmatrix} -1 & 1 & 2 \\ 1 & -1 & 0 \\ 2 & 0 & -1 \end{bmatrix} \tag{10}$$

as well as the generalized identity matrix

$$I_g = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{11}$$

The system zeroes are given by the finite generalized egienvalues to the following generalized eigenvalue/eigenvector problem

$$SM = I_g M \Lambda. \tag{12}$$

This problem has three generalized eigenvectors m_1, m_2 and m_3. These are the columns in the generalized eigenvector matrix $M = \begin{bmatrix} m_1 & m_2 & m_3 \end{bmatrix}$. Λ is a diagonal matrix with the generalized eigenvalues λ_1, λ_2 and λ_3 on the diagonal. note that the eigenvalues may be complex as well as real.
We define

$$m_1 = \begin{bmatrix} m_{11} \\ m_{21} \\ m_{31} \end{bmatrix}. \tag{13}$$

Ve have the following problem

$$Sm_1 = I_g m_1 \lambda_1 \tag{14}$$

for the definition of m_1 and λ_1. This gives

$$\begin{bmatrix} -1 & 1 & 2 \\ 1 & -1 & 0 \\ 2 & 0 & -1 \end{bmatrix} \begin{bmatrix} m_{11} \\ m_{21} \\ m_{31} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} m_{11} \\ m_{21} \\ m_{31} \end{bmatrix} \lambda_1 \tag{15}$$

This gives

$$-m_{11} + m_{21} + 2m_{31} = m_{11}\lambda_1, \tag{16}$$
$$m_{11} - m_{21} = 0, \tag{17}$$
$$2m_{11} - m_{31} = 0. \tag{18}$$

The two last equationes gives $m_{31} = 2m_{11}$ and $m_{21} = m_{11}$.
substituting this into the first equation gives

$$-m_{11} + m_{11} + 4m_{11} = m_{11}\lambda_1. \tag{19}$$

this equation holds for all m_{11} different from zero.
Hence, $\lambda_1 = 4$.
The system have then a zero $s = 4$.
The system has only one finite zero. Notice that the generalized eigenvalue problem also have two generalized eigenvalues at infinity, i.e., $\lambda_2 = \infty$ and $\lambda_3 = \infty$.

Task 4

The system have a zero

$$s_0 = -\frac{e+1}{e}. \tag{20}$$

The system has a zero for all $e \neq 0$. This is shown from the transfer function

$$h(s) = \frac{es+e+1}{s+1}. \tag{21}$$

The system has a positive zero for all $-1 < e < 0$. The system is non-minimum pase in this case The system have a zero in the right half plane in this case.

Task 5

a) The transfer matrix $H(s)$ is an $m \times r$ matrix where $m = 3$ and $r = 2$ in this case.

   The system have $r = 2$ control inputs and $m = 3$ outputs. The rank of the transfer matrix $H(s)$ can maximum be min(m,r) = 2. This is defined as the natural rank of $H(s)$.

   Notice that there may be values $s = z$ such that $\text{rang}(H(z)) < \min(m, r)$.

   such a value for s is defined for a transmission zero for the MIMO system.

b) The system have a zero $s = 2$. We have

$$H(2) = \begin{bmatrix} \frac{1}{12} & -\frac{1}{12} \\ \frac{1}{6} & -\frac{1}{6} \\ 0 & 0 \end{bmatrix}. \tag{22}$$

c) We see that rank$(H(2)) = 1$ because column number one in $H(2)$ is equal to column number two.

d) A Singular Value Decomposition (SVD) of $H(2)$ is given by

$$H(2) = USV^T = \sum_{i=1}^{\min(m,r)} u_i s_i v_i^T = u_1 s_1 v_1^T + u_2 s_2 v_2^T, \tag{23}$$

   where

$$U = \begin{bmatrix} u_1 & u_2 & u_3 \end{bmatrix} = \begin{bmatrix} 0.4472 & -0.8944 & 0 \\ 0.8944 & 0.4472 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{24}$$

$$S = \begin{bmatrix} s_1 & 0 \\ 0 & s_2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0.2635 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \tag{25}$$

   and

$$V = \begin{bmatrix} v_1 & v_2 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix}. \tag{26}$$

We may find u_z as the right singular vector to the zero singular value. This will in this case be $u_z = v_2$ where v_2 is column number two in V where $V = \left[\begin{array}{cc} v_1 & v_2 \end{array} \right]$. This gives

$$u_z = \left[\begin{array}{c} -\frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} \end{array} \right]. \tag{27}$$

Such a control input results in $y = Gu_z = 0$.

This is shown in the following

$H(2) = u_1 s_1 v_1^T + u_2 s_2 v_2^T$ with $u_z = v_2$. This gives

$$H(2)v_2 = u_1 s_1 v_1^T v_2 + u_2 s_2 v_2^T v_2, \tag{28}$$

and use that V is an orthogonal matrix, i,e., such that $V^T V = I$, $v_1^T v_2 = 0$ and $v_2^T v_2 = 1$. Ve have that

$$H(2)v_2 = u_2 s_2. \tag{29}$$

Then $s_2 = 0$. $u_z = v_2$ is therefore the control input resulting in a zero output.

Notice: We have $\min(m, r) = 2$ singular values, i.e.. $s_1 = 0.2635$ and $s_2 = 0$. These values are identical to the diagonal elements in S. This gives $\text{rang}(H(z)) = 1$ because there only is one singular value different from zero.