Master study
IIAV3017
Faculty of Technology
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IIAV3017 Advanced Control

Solution Exercise 1

The system can be written in matrix state space form as follows

$$\dot{x} = \overbrace{\begin{bmatrix} -5 & -2 \\ 0 & -1 \end{bmatrix}}^{A} x + \overbrace{\begin{bmatrix} \beta \\ 1 \end{bmatrix}}^{B} u, \tag{1}$$

$$y = \underbrace{\begin{bmatrix} 1 & \delta \end{bmatrix}}_{D} x. \tag{2}$$

where $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

Solution task 1b)

The controllability matrix for this system is given by

$$C_2 = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} \beta & -5\beta - 2 \\ 1 & -1 \end{bmatrix} \tag{3}$$

The system is controllable if $\text{rank}(C_2) = 2$. This can be investigated in at least two ways.
Method 1 No columns ore rows in C_n should be linearly dependent in order for the controllability matrix to have full rank, i.e., $\text{rank}(C_2) = 2$ since $n = 2$ in this example. This means that there should not exists a constant k such that

$$\begin{bmatrix} \beta \\ 1 \end{bmatrix} = k \begin{bmatrix} -5\beta - 2 \\ -1 \end{bmatrix} \tag{4}$$

Since there are two unknowns in the problem we chose k, e.g., in such a way that the left and right hand sides becomes as close as possible. $k = -1$ gives

$$\begin{bmatrix} \beta \\ 1 \end{bmatrix} = \begin{bmatrix} 5\beta + 2 \\ 1 \end{bmatrix} \tag{5}$$

We see that the two vectors becomes equal when $5\beta + 2 = \beta$, and that the two vectors always is different, i.e. linearly independent, when

$$5\beta + 2 \neq \beta \tag{6}$$

which shows that the system is controllable if $\beta \neq -\frac{1}{2}$.
Method 2 A simple method is to investigate the determinant of C_2. The system is controllable if

$$\det(C_2) = -\beta + 5\beta + 2 \neq 0 \tag{7}$$

This gives that $\beta \neq -\frac{1}{2}$ for the system to be controllable.
Note that the demand $\mathrm{rank}(C_n) = n$ only is equivalent to investigate if $\det(C_n) \neq 0$ for systems with a single control input variable, i.e. for $r = 1$. The determinant method can not be used for controllability analysis for systems with multiple control input variables, i.e. when $r > 1$, because C_n is not a quadratic matrix in this case.

Solution task 1c)

The observability matrix of the system is given by

$$O_2 = \begin{bmatrix} D \\ DA \end{bmatrix} = \begin{bmatrix} 1 & \delta \\ -5 & -2-\delta \end{bmatrix} \tag{8}$$

The system is observable if $\mathrm{rank}(O_2) = 2$ which for systems with one output variable is equivalent to check wether

$$\det(O_2) = -2 - \delta + 5\delta \neq 0 \tag{9}$$

This means that the system is observable if $\delta \neq \frac{1}{2}$.

Solution task 1d)

The diagonal form of a state space model $\dot{(x)} = Ax + Bu$, $y = Dx$, may be found by using a transformation $x = Mz$ where M is the eigenvector matrix to system matrix A and z is the state vector the diagonal (state space) form. This gives

$$\dot{z} = \Lambda z + M^{-1}Bu, \tag{10}$$
$$y = DMz, \tag{11}$$

where Λ is the diagonal eigenvalue matrix of the system

$$\Lambda = M^{-1}AM = \begin{bmatrix} -5 & 0 \\ 0 & -1 \end{bmatrix}. \tag{12}$$

An eigenvector matrix M (and the inverse) for this system is given by

$$M = \begin{bmatrix} 1 & 1 \\ 0 & -2 \end{bmatrix}, M^{-1} = \begin{bmatrix} 1 & \frac{1}{2} \\ 0 & -\frac{1}{2} \end{bmatrix} \tag{13}$$

We may now analyze controllability by studying the elements in the transformed matrix $M^{-1}B$ and we may study observability by checking the elements in the matrix DM.

The system is controllable if no row or rows in the matrix $M^{-1}B$ have all elements identically equal to zero. We have

$$M^{-1}B = \begin{bmatrix} \beta + \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}. \tag{14}$$

We se that the system is controllable if now columns in the matrix DM have all elements equal to zero.
We find that the system is controllable if $\beta \neq -\frac{1}{2}$
Furthermore we have that the system is observable if no columns in the matrix DM have all elements equal to zero. We have that

$$DM = \begin{bmatrix} 1 & 1-2\delta \end{bmatrix}. \tag{15}$$

As we se tha system is observable if $1 - 2\delta \neq 0$, i.e. observable for $\delta = \frac{1}{2}$.

Solution to task 1e)

Given the system

$$\begin{aligned} \dot{x}_1 &= -5x_1 - 2x_2 + \beta u && (16)\\ \dot{x}_2 &= -x_2 + u && (17)\\ y &= x_1 + \delta x_2 && (18) \end{aligned}$$

We have the following state space model

$$\begin{aligned} \dot{x} &= Ax + Bu && (19)\\ y &= Dx + Eu && (20) \end{aligned}$$

where the system matrices are given by

$$\begin{aligned} A &= \begin{bmatrix} -5 & -2 \\ 0 & -1 \end{bmatrix} \quad B = \begin{bmatrix} \beta \\ 1 \end{bmatrix} && (21)\\ D &= \begin{bmatrix} 1 & \delta \end{bmatrix} \quad E = 0 && (22) \end{aligned}$$

Controllability Gramian matrix
The system is stable because A has eigenvalues with negative real part, i.e., $\lambda_1 = -5$ and $\lambda_2 = -1$). The infinity time controllability Gramian matrix, W_c, then satisfy the Lyapunov matrix equation

$$AW_c + W_c A^T = -BB^T \tag{23}$$

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$$AW_c = \begin{bmatrix} -5 & -2 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} w_{11} & w_{21} \\ w_{21} & w_{22} \end{bmatrix} = \begin{bmatrix} -5w_{11} - 2w_{21} & -5w_{21} - 2w_{22} \\ -w_{21} & -w_{22} \end{bmatrix} \tag{24}$$

$$W_c A^T = (AW_c)^T = \begin{bmatrix} -5w_{11} - 2w_{21} & -w_{21} \\ -5w_{21} - 2w_{22} & -w_{22} \end{bmatrix} \tag{25}$$

$$BB^T = \begin{bmatrix} \beta \\ 1 \end{bmatrix} \begin{bmatrix} \beta & 1 \end{bmatrix} = \begin{bmatrix} \beta^2 & \beta \\ \beta & 1 \end{bmatrix} \tag{26}$$

Putting (24), (25) and (26) in the Lyapunov equation (23) and we get the following three equations for the unknown elements in the matrix W_c, i.e.,

$$\begin{aligned} -10w_{11} - 4w_{21} &= -\beta^2 && (27) \\ -6w_{21} - 2w_{22} &= -\beta && (28) \\ -2w_{22} &= -1 && (29) \end{aligned}$$

This gives the controllability Gramian matrix

$$W_c = \begin{bmatrix} w_{11} & w_{21} \\ w_{21} & w_{22} \end{bmatrix} = \begin{bmatrix} \frac{1}{10}(\beta^2 - \frac{2}{3}\beta + \frac{2}{3}) & \frac{1}{6}(\beta - 1) \\ \frac{1}{6}(\beta - 1) & \frac{1}{2} \end{bmatrix} \tag{30}$$

In order for the system to be controllable we have to ensure that W_c is positive definite, i.e.,

$$W_c > 0 \tag{31}$$

This is ensured if the eigenvalues of W_c is all positive or if all sub determinants of W_c is positive. We have the following requirements (positive sub determinants of order 1 and 2)

$$\begin{aligned} w_{11} &= \frac{1}{10}(\beta^2 - \frac{2}{3}\beta + \frac{2}{3}) > 0 && (32) \\ det(W_c) &= w_{11}w_{22} - w_{21}^2 = \frac{1}{45}(\beta^2 + \beta + \frac{1}{4}) > 0 && (33) \end{aligned}$$

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$$det(W_c) > 0 \quad \text{for} \quad \beta \neq -\frac{1}{2} \tag{34}$$

$w_{11} > 0$ for all real β.
We se that $w_{11} \to \infty$ når $\beta \to \infty$ and $\beta \to -\infty$. Furthermore we have that w_{11} have a minimum for $\frac{dw_{11}}{d\beta} = \frac{1}{3}$. This means that $w_{11} > 0$ for all real β. Note however that $w_{11} = 0$ for $\beta = \frac{1}{3} \pm \frac{\sqrt{5}}{3}$.
The conclusion is that the system is controllable for all $\beta \neq -\frac{1}{2}$.

Observability Gramian matrix
We solve the Lyapunov matrix equation

$$A^T W_o + W_o A = -D^T D \tag{35}$$

with respect to the observability Gramian matrix W_o. We have that

$$A^T W_o = \begin{bmatrix} -5 & 0 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} w_{11} & w_{21} \\ w_{21} & w_{22} \end{bmatrix} = \begin{bmatrix} -5w_{11} & -5w_{21} \\ -2w_{11} - w_{21} & -2w_{21} - w_{22} \end{bmatrix} \tag{36}$$

$$W_o A = (A^T W_o)^T = \begin{bmatrix} -5w_{11} & -2w_{11} - w_{21} \\ -5w_{21} & -2w_{21} - w_{22} \end{bmatrix} \tag{37}$$

$$D^T D = \begin{bmatrix} 1 \\ \delta \end{bmatrix} \begin{bmatrix} 1 & \delta \end{bmatrix} = \begin{bmatrix} 1 & \delta \\ \delta & \delta^2 \end{bmatrix} \tag{38}$$

This gives three equations

$$\begin{aligned} -10w_{11} &= -1 && (39) \\ -2w_{11} - 6w_{21} &= -\delta && (40) \\ -4w_{21} - 2w_{22} &= -\delta^2 && (41) \end{aligned}$$

which may be solved with respect to w_{11}, w_{21} and w_{22}. This gives

$$w_{11} = \frac{1}{10} \tag{42}$$

$$w_{21} = \frac{1}{6}(\delta - \frac{1}{5}) \tag{43}$$

$$w_{22} = \frac{1}{2}(\delta^2 - \frac{2}{3}\delta + \frac{4}{30}) \tag{44}$$

This gives the observability Gramian matrix

$$W_o = \begin{bmatrix} w_{11} & w_{21} \\ w_{21} & w_{22} \end{bmatrix} = \begin{bmatrix} \frac{1}{10} & \frac{1}{6}(\delta - \frac{1}{5}) \\ \frac{1}{6}(\delta - \frac{1}{5}) & \frac{1}{2}(\delta^2 - \frac{2}{3}\delta + \frac{4}{30}) \end{bmatrix} \tag{45}$$

We investigate if W_o may have rank less than two for some values of parameter δ. We have

$$\begin{aligned} \det(W_o) &= w_{11}w_{22} - w_{21}^2 = \frac{1}{20}(\delta^2 - \frac{2}{3}\delta + \frac{4}{30}) - \frac{1}{36}(\delta - \frac{1}{5})^2 && (46) \\ &= \frac{1}{45}(\delta^2 - \delta + \frac{1}{4}) = \frac{1}{45}(\delta - \frac{1}{2})^2 && (47) \end{aligned}$$

We se that $\det(W_o) = 0$ for $\delta = \frac{1}{2}$. W_o is therefore singular (not invertable) for this value of δ. This means that the system is not observable for $\delta = \frac{1}{2}$.

Solution of Task 2

a) We try with a vector

$$b = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{48}$$

The controllability matrices and the coefficients in the characteristic polynomial are

$$p(A) = \lambda^4 + p_4\lambda^3 + p_3\lambda^2 + p_2\lambda + p_1 \tag{49}$$

For the two cases we have
Case 1

$$C_4 = \begin{bmatrix} b & Ab & A^2b & A^3b \end{bmatrix} = \begin{bmatrix} 1 & -5 & 23 & -65 \\ 0 & 0 & 12 & -48 \\ 0 & 2 & 2 & -22 \\ 0 & 1 & -1 & -5 \end{bmatrix} \tag{50}$$

and

$$p = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{bmatrix} = -C_4^{-1}A^4b = \begin{bmatrix} 0 \\ -2 \\ -1 \\ 2 \end{bmatrix} \tag{51}$$

Case 2

$$C_4 = \begin{bmatrix} b & Ab & A^2b & A^3b \end{bmatrix} = \begin{bmatrix} 1 & -3 & -1 & 11 \\ 0 & 10 & -8 & -6 \\ 0 & -12 & 20 & -24 \\ 0 & 4 & -8 & 12 \end{bmatrix} \tag{52}$$

It tursn out that for this choice of b then C_4 may not be inverted, this means that the pair (A, b) is not controllable because $\text{rank}(C_4) = 3 < n = 4$, However chosing

$$b = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \tag{53}$$

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$$C_4 = \begin{bmatrix} b & Ab & A^2b & A^3b \end{bmatrix} = \begin{bmatrix} 0 & -1 & -14 & 51 \\ 1 & 17 & -9 & -37 \\ 0 & -32 & 64 & -92 \\ 0 & 12 & -28 & 48 \end{bmatrix} \tag{54}$$

and

$$p = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{bmatrix} = -C_4^{-1} A^4 b = \begin{bmatrix} 1 \\ 4 \\ 6 \\ 4 \end{bmatrix} \tag{55}$$

b) A controllability canonical form is obtained by transforming the model with

$$z = C_4^{-1} x \tag{56}$$

i.e.,

$$\dot{z} = A_{co} z \tag{57}$$

where

$$A_{co} = C_4^{-1} A C_4 \tag{58}$$

with initial state $z(t=0) = C_4^{-1} x(t=0)$.
case 1 The system matrix on controllability canonical form is in this case

$$A_{co} = \begin{bmatrix} 0 & 0 & 0 & -p_1 \\ 1 & 0 & 0 & -p_2 \\ 0 & 1 & 0 & -p_3 \\ 0 & 0 & 1 & -p_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix} \tag{59}$$

Case 2 The system matrix on controllability canonical form is in this case

$$A_{co} = \begin{bmatrix} 0 & 0 & 0 & -p_1 \\ 1 & 0 & 0 & -p_2 \\ 0 & 1 & 0 & -p_3 \\ 0 & 0 & 1 & -p_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & -4 \\ 0 & 1 & 0 & -6 \\ 0 & 0 & 1 & -4 \end{bmatrix} \tag{60}$$

Note that the controllability canonical form may be computed by first generating the controllability matrix

$$C_{n+1} = \begin{bmatrix} b & Ab & \cdots & A^{n-1}b & A^n b \end{bmatrix} \tag{61}$$

and thereafter compute

$$A_{co} = -C_{n+1}^{-1}(:, 1:n) C_{n+1}(:, 2:n+1; \tag{62}$$