University of South-Eastern Norway

EXAMINATION INFORMATION PAGE
Written examination

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Responsible subject teacher: David Di Ruscio		
Campus: Porsgrunn	Faculty: Faculty of Technology	
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Comments:		

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Task 1 (30%) (Diverse questions)

a) Given a system described by the continuous linear state space model

$$\dot{x} = Ax + Bu. \tag{1}$$

Consider the LQ optimal control objective,

$$J = \frac{1}{2}\int_{t_0}^{t_1} [x^T Q x + u^T P u] dt, \tag{2}$$

where t_0 is equal to the present time t and where the final time instant t_1 is equal to $t+T$ where $T > 0$ is a constant prediction horizon. Hence, we are putting $t_0 = t$ and $t_1 = t + T$ in the objective Eq. (2).

From the maximum principle as presented in the enclosed Attachment we may deduce the autonomous system

$$\dot{\tilde{x}} = F\tilde{x} \tag{3}$$

where

$$\tilde{x} = \begin{bmatrix} x \\ p \end{bmatrix}, \quad \dot{\tilde{x}} = \begin{bmatrix} \dot{x} \\ \dot{p} \end{bmatrix}. \tag{4}$$

Answer the following:

1. Find the system matrix F.
2. Find en expression for R in the linear relationship $p = Rx$.
3. Is R a constant matrix or time varying in this case? Comment upon the answer !

Hints: You can use that the solution to the autonomous system (3) is given by

$$\tilde{x}(t) = e^{F(t-t_0)}\tilde{x}(t_0) = \Phi\tilde{x}(t_0) \tag{5}$$

where we also can partition the transition matrix $\Phi = e^{F(t-t_0)}$ as follows

$$\Phi = \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{bmatrix}. \tag{6}$$

We may express R as a function of the sub-matrices in Φ and S.

Remarks : R is the solution to the Riccati equation of this problem.

b) What is meant with the *Separation Principle* ?

c) *Explain the principle of duality between optimal control and Optimal estimation and write up the duality table ?*

d) *What is a LQG controller ? Comment and sketch a block diagram !*

e) *Is optimal control in general: of state feedback type ? or of output feedback type ?*

Task 2 (24%) (Continuous time optimal control)

Given an LQ optimal control objective defined over a finite time horizon, i.e.,

$$J = \frac{1}{2}x(t_1)^T S x(t_1) + \frac{1}{2}\int_{t_0}^{t_1} [y^T Q y + u^T P u] dt, \tag{7}$$

where $S \in \mathbb{R}^{n\times n}$, $Q \in \mathbb{R}^{m\times m}$ and $P \in \mathbb{R}^{r\times r}$ are symmetric weighting matrices.

a) Consider a strictly proper continuous time state space model

$$\dot{x} = Ax + Bu, \tag{8}$$
$$y = Dx, \tag{9}$$

for a process. Find the optimal control which minimize the objective, J, given by (7) subject to the model (8) and (9).

The solution should consist of:

1. An expression for the optimal control vector, u^*.
2. A Riccati equation.
3. A final value condition for the Riccati equation.

b) Consider now an infinite horizon LQ control objective as follows

$$J = \frac{1}{2}\int_{0}^{\infty} [y^T Q y + u^T P u] dt, \tag{10}$$

and the model as in (8) and (9) above.

What is now the solution to the LQ optimal control problem?

c) How can we ensure stability of the controlled feedback system in Task 2b) above ? The answer should involve the model matrices A and B as well as the weighting matrices Q and P.

d) Consider a proper continuous time state space model

$$\dot{x} = Ax + Bu, \tag{11}$$
$$y = Dx + Eu, \tag{12}$$

for a process. Find the optimal control which minimize the objective, J, given by (10) subject to the model (11) and (12).

Task 3 (18%) (Discrete time optimal control)

Given an LQ optimal criterion defined over the time interval $i \leq k \leq \infty$, i.e.,

$$J_i = \frac{1}{2}\sum_{k=i}^{\infty}[y_k^T Q y_k + u_k^T P u_k], \tag{13}$$

where $Q \in \mathbb{R}^{m \times m}$ and $P \in \mathbb{R}^{r \times r}$ are symmetric weighting matrices.

a) Assume that the system is modeled with a discrete time linear state space model, i.e.,

$$x_{k+1} = Ax_k + Bu_k, \tag{14}$$
$$y_k = Dx_k. \tag{15}$$

Find the optimal controller which are minimizing the objective J_i given by (13) and subject to the model (14) and (15)

The solution should consist of, An expression for the optimal control u_k^* and a discrete Riccati equation.

b) Consider now an objective function with cross term in the objective function, i.e.

$$J_i = \frac{1}{2}\sum_{k=i}^{\infty}[x_k^T Q x_k + 2x_k^T N u_k + u_k^T P u_k], \tag{16}$$

Find the optimal controller which are minimizing the objective J_i given by Eq. (16) and subject to the state equation Eq. (14) !

The solution should consist of, an expression for the optimal control u_k^* and a Discrete Algebraic Riccati Equation (DARE).

c) Is there a connection with the problem in 3b) above and the problem of minimizing the objective (13) subject to Eq. (14) with an output equation with direct feed trough term as

$$y_k = Dx_k + Eu_k \tag{17}$$

? Comment upon some details upon the solution ?

Task 4 (28%) (LQ optimal control with Integral Action)

We are in this task to study an LQ optimal controller for a system described by the state space model

$$x_{k+1} = Ax_k + Bu_k + v, \tag{18}$$
$$y_k = Dx_k + w, \tag{19}$$

where v and w are constant and unknown disturbances.
Subject to the above state space model we want to design an LQ optimal controller which minimizes the following LQ criterion

$$J_i = \frac{1}{2}\sum_{k=i}^{\infty}((r-y_k)^T Q(r-y_k) + \Delta u_k^T P \Delta u_k). \tag{20}$$

where $\Delta u_k = u_k - u_{k-1}$ and r is a constant reference vector. Q and P are symmetric and positive semidefinite matrices.

a) Show that it is possible to write the model in (18) and (19) on deviation form, i.e.,

$$\Delta x_{k+1} = A\Delta x_k + B\Delta u_k, \tag{21}$$
$$\Delta y_k = D\Delta x_k, \tag{22}$$

where

$$\Delta x_k = x_k - x_{k-1}, \ \Delta u_k = u_k - u_{k-1}, \ \Delta y_k = y_k - y_{k-1}. \tag{23}$$

What can be gained by doing this?

b) Show that the model in (21) and (22) can be written as follows

$$\tilde{x}_{k+1} = \tilde{A}\tilde{x}_k + \tilde{B}\Delta u_k, \tag{24}$$
$$\tilde{y}_k = \tilde{D}\tilde{x}_k, \tag{25}$$

where

$$\tilde{x}_k = \begin{bmatrix} \Delta x_k \\ r - y_{k-1} \end{bmatrix}, \ \tilde{y}_k = r - y_k. \tag{26}$$

Here you should define the matrices $\tilde{A}$, $\tilde{B}$ and $\tilde{D}$.

c) The state space model in 4b) and the LQ criterion in (20) defines a standard discrete LQ optimal control problem of the form

$$\tilde{x}_{k+1} = \tilde{A}\tilde{x}_k + \tilde{B}\tilde{u}_k, \tag{27}$$
$$\tilde{y}_k = \tilde{D}\tilde{x}_k, \tag{28}$$

with LQ criterion

$$J_i = \frac{1}{2}\sum_{k=i}^{\infty}(\tilde{y}_k^T Q\tilde{y}_k + \tilde{u}_k^T P\tilde{u}_k), \tag{29}$$

where we for simplicity has defined

$$\tilde{u}_k = \Delta u_k. \tag{30}$$

Write down the solution to the LQ optimal control problem, of the form

$$\tilde{u}_k^* = \tilde{G}\tilde{x}_k. \tag{31}$$

the solution should consist of:

1. A discrete Riccati equation
2. an expression for the controller matrix $\tilde{G}$.
3. Write down an expression for the actual control of the form

$$u_k = f(\cdot) \tag{32}$$

   which are to be used in order to control the process.

d) Comment upon possible similarities with the above deduced optimal controller in Task 4c) and a conventional PI controller ?

Attachment

Continuous time optimal control

$$\dot{x} = f(x, u, t) \tag{33}$$

$$J = S(x(t_1)) + \int_{t_0}^{t_1} L(x, u, t)dt \tag{34}$$

The continuous time Maximum Principle

$$H = L + p^T f \tag{35}$$

$$\dot{p} = -\frac{\partial H}{\partial x} \quad p(t_1) = \frac{\partial S}{\partial x}|_{t_1} \tag{36}$$

$$\frac{\partial H}{\partial u} = 0. \tag{37}$$

Discrete time optimal control

$$x_{k+1} - x_k = f(x_k, u_k, k) \tag{38}$$

$$J_i = S(x_N) + \sum_{k=i}^{N-1} L(x_k, u_k, k) \tag{39}$$

The discrete time maximum Principle

$$H_k = L(x_k, u_k, k) + p_{k+1}^T (x_{k+1} - x_k) \tag{40}$$

$$p_{k+1} - p_k = -\frac{\partial H_k}{\partial x_k}, \quad p_N = \frac{\partial S}{\partial x_N}, \tag{41}$$

$$\frac{\partial H_k}{\partial u_k} = 0. \tag{42}$$

Derivation rules

$$\frac{\partial}{\partial x}(Qx) = Q^T, \quad \frac{\partial}{\partial x}(x^T Q x) = Qx + Q^T x, \tag{43}$$

$$\frac{\partial}{\partial x}((r - Dx)^T Q (r - Dx)) = -2D^T Q (r - Dx). \tag{44}$$