

EXAMINATION INFORMATION PAGE

Written examination

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| Subject code: IIAV3017 | Subject name: Advanced Control | |
| Examination date: November 26 th 2018 | Examination time from/to: 9-14 | Total hours: 5 |
| Responsible subject teacher: David Di Ruscio | | |
| Campus: Porsgrunn | Faculty: Faculty of Technology | |
| No. of assignments: 4 | No. of attachments: 1 | No. of pages incl. front page and attachments: 7 |
| Permitted aids: Pen and paper | | |
| Information regarding attachments: The Maximum principle for continuous and discrete time optimal control plus derivation rules. | | |
| Comments: | | |

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| Select the type of examination paper | Spreadsheets <input type="checkbox"/> | Line sheets <input type="checkbox"/> |
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Task 1 (25%) (Diverse questions)

a) Given a system described by the continuous linear state space model

$$\dot{x} = Ax + Bu. \quad (1)$$

Consider the LQ optimal control objective defined over a time horizon $t_0 \leq t \leq t_1$ i.e.,

$$J = \frac{1}{2} \int_{t_0}^{t_1} [x^T Q x + u^T P u] dt, \quad (2)$$

where t_0 is equal to the present time t and where the final time instant t_1 is equal to $t+T$ where $T > 0$ is a constant prediction horizon. Hence, we are putting $t_0 = t$ and $t_1 = t+T$ in the objective Eq. (2). Furthermore, $Q \in \mathbb{R}^{m \times m}$ and $P \in \mathbb{R}^{r \times r}$ are prescribed symmetric weighting matrices.

From the maximum principle as presented in the enclosed Attachment we may deduce the autonomous system

$$\dot{\tilde{x}} = F \tilde{x} \quad (3)$$

where

$$\tilde{x} = \begin{bmatrix} x \\ p \end{bmatrix}, \quad \dot{\tilde{x}} = \begin{bmatrix} \dot{x} \\ \dot{p} \end{bmatrix}. \quad (4)$$

Answer the following:

1. Find the system matrix F .
2. Find an expression for R in the linear relationship $p = Rx$.
3. Is R a constant matrix or time varying in this case? Comment upon the answer !

Hints: You can use that the solution to the autonomous system (3) is given by

$$\tilde{x}(t) = e^{F(t-t_0)} \tilde{x}(t_0) = \Phi \tilde{x}(t_0) \quad (5)$$

where we also can partition the transition matrix $\Phi = e^{F(t-t_0)}$ as follows

$$\Phi = \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{bmatrix}. \quad (6)$$

We may express R as a function of the sub-matrices in Φ and S .

Remarks : R is the solution to the Riccati equation of this problem.

- b) What is meant with the *Separation Principle* ?
- c) Explain the principle of duality between optimal control and Optimal estimation and write up the duality table ?
- d) What is a LQG controller ? Comment and sketch a block diagram !
- e) Is optimal control in general: of state feedback type ? or of output feedback type ?

Task 2 (20%) (Continuous time optimal control)

Given an LQ optimal control objective defined over a finite time horizon, i.e.,

$$J = \frac{1}{2}x(t_1)^T Sx(t_1) + \frac{1}{2} \int_{t_0}^{t_1} [y^T Qy + u^T Pu]dt, \quad (7)$$

where $S \in \mathbb{R}^{n \times n}$, $Q \in \mathbb{R}^{m \times m}$ and $P \in \mathbb{R}^{r \times r}$ are symmetric weighting matrices.

- a) Consider a strictly proper continuous time state space model

$$\dot{x} = Ax + Bu, \quad (8)$$

$$y = Dx, \quad (9)$$

for a process. Find the optimal control which minimize the objective, J , given by (7) subject to the model (8) and (9).

The solution should consist of:

1. An expression for the optimal control vector, u^* .
2. A Riccati equation.
3. A final value condition for the Riccati equation.

- b) Consider now an infinite horizon LQ control objective as follows

$$J = \frac{1}{2} \int_0^{\infty} [y^T Qy + u^T Pu]dt, \quad (10)$$

and the model as in (8) and (9) above.

What is now the solution to the LQ optimal control problem?

- c) How can we ensure stability of the controlled feedback system in Task 2b) above ? The answer should involve the model matrices A and B as well as the weighting matrices Q and P .

d) Consider a proper continuous time state space model

$$\dot{x} = Ax + Bu, \quad (11)$$

$$y = Dx + Eu, \quad (12)$$

for a process. Find the optimal control which minimize the objective, J , given by (10) subject to the model (11) and (12).

Task 3 (15%) (Discrete time optimal control)

Given an LQ optimal criterion defined over the time interval $i \leq k \leq \infty$, i.e.,

$$J_i = \frac{1}{2} \sum_{k=i}^{\infty} [y_k^T Q y_k + u_k^T P u_k], \quad (13)$$

where $Q \in \mathbb{R}^{m \times m}$ and $P \in \mathbb{R}^{r \times r}$ are symmetric weighting matrices.

a) Assume that the system is modeled with a discrete time linear state space model, i.e.,

$$x_{k+1} = Ax_k + Bu_k, \quad (14)$$

$$y_k = Dx_k. \quad (15)$$

Find the optimal controller which are minimizing the objective J_i given by (13) and subject to the model (14) and (15)

The solution should consist of, An expression for the optimal control u_k^* and a discrete Riccati equation.

b) Consider now an objective function with cross term in the objective function, i.e.

$$J_i = \frac{1}{2} \sum_{k=i}^{\infty} [x_k^T Q x_k + 2x_k^T N u_k + u_k^T P u_k], \quad (16)$$

Find the optimal controller which are minimizing the objective J_i given by Eq. (16) and subject to the state equation Eq. (14) !

The solution should consist of, an expression for the optimal control u_k^* and a Discrete Algebraic Riccati Equation (DARE).

c) Is there a connection with the problem in 3b) above and the problem of minimizing the objective (13) subject to Eq. (14) with an output equation with direct feed through term as

$$y_k = Dx_k + Eu_k \quad (17)$$

? Comment upon some details upon the solution ?

Task 4 (20%) (LQ optimal control with Integral Action)

We are in this task to study an LQ optimal controller for a system described by the state space model

$$x_{k+1} = Ax_k + Bu_k + v, \quad (18)$$

$$y_k = Dx_k + w, \quad (19)$$

where v and w are constant and unknown disturbances.

Subject to the above state space model we want to design an LQ optimal controller which minimizes the following LQ criterion

$$J_i = \frac{1}{2} \sum_{k=i}^{\infty} ((r - y_k)^T Q (r - y_k) + \Delta u_k^T P \Delta u_k). \quad (20)$$

where $\Delta u_k = u_k - u_{k-1}$ and r is a constant reference vector. Q and P are symmetric and positive semidefinite matrices.

- a) Show that it is possible to write the model in (18) and (19) on deviation form, i.e.,

$$\Delta x_{k+1} = A\Delta x_k + B\Delta u_k, \quad (21)$$

$$\Delta y_k = D\Delta x_k, \quad (22)$$

where

$$\Delta x_k = x_k - x_{k-1}, \quad \Delta u_k = u_k - u_{k-1}, \quad \Delta y_k = y_k - y_{k-1}. \quad (23)$$

What can be gained by doing this?

- b) Show that the model in (21) and (22) can be written as follows

$$\tilde{x}_{k+1} = \tilde{A}\tilde{x}_k + \tilde{B}\Delta u_k, \quad (24)$$

$$\tilde{y}_k = \tilde{D}\tilde{x}_k, \quad (25)$$

where

$$\tilde{x}_k = \begin{bmatrix} \Delta x_k \\ r - y_{k-1} \end{bmatrix}, \quad \tilde{y}_k = r - y_k. \quad (26)$$

Here you should define the matrices \tilde{A} , \tilde{B} and \tilde{D} .

c) The state space model in 4b) and the LQ criterion in (20) defines a standard discrete LQ optimal control problem of the form

$$\tilde{x}_{k+1} = \tilde{A}\tilde{x}_k + \tilde{B}\tilde{u}_k, \quad (27)$$

$$\tilde{y}_k = \tilde{D}\tilde{x}_k, \quad (28)$$

with LQ criterion

$$J_i = \frac{1}{2} \sum_{k=i}^{\infty} (\tilde{y}_k^T Q \tilde{y}_k + \tilde{u}_k^T P \tilde{u}_k), \quad (29)$$

where we for simplicity has defined

$$\tilde{u}_k = \Delta u_k. \quad (30)$$

Write down the solution to the LQ optimal control problem, of the form

$$\tilde{u}_k^* = \tilde{G}\tilde{x}_k. \quad (31)$$

the solution should consist of:

1. A discrete Riccati equation
2. an expression for the controller matrix \tilde{G} .
3. Write down an expression for the actual control of the form

$$u_k = f(\cdot) \quad (32)$$

which are to be used in order to control the process.

d) Comment upon possible similarities with the above deduced optimal controller in Task 4c) and a conventional PI controller ?

Attachment

Continuous time optimal control

$$\dot{x} = f(x, u, t) \quad (33)$$

$$J = S(x(t_1)) + \int_{t_0}^{t_1} L(x, u, t) dt \quad (34)$$

The continuous time Maximum Principle

$$H = L + p^T f \quad (35)$$

$$\dot{p} = -\frac{\partial H}{\partial x} p(t_1) = \frac{\partial S}{\partial x} \Big|_{t_1} \quad (36)$$

$$\frac{\partial H}{\partial u} = 0. \quad (37)$$

Discrete time optimal control

$$x_{k+1} - x_k = f(x_k, u_k, k) \quad (38)$$

$$J_i = S(x_N) + \sum_{k=i}^{N-1} L(x_k, u_k, k) \quad (39)$$

The discrete time maximum Principle

$$H_k = L(x_k, u_k, k) + p_{k+1}^T (x_{k+1} - x_k) \quad (40)$$

$$p_{k+1} - p_k = -\frac{\partial H_k}{\partial x_k}, \quad p_N = \frac{\partial S}{\partial x_N}, \quad (41)$$

$$\frac{\partial H_k}{\partial u_k} = 0. \quad (42)$$

Derivation rules

$$\frac{\partial}{\partial x} (Qx) = Q^T, \quad \frac{\partial}{\partial x} (x^T Qx) = Qx + Q^T x, \quad (43)$$

$$\frac{\partial}{\partial x} ((r - Dx)^T Q(r - Dx)) = -2D^T Q(r - Dx). \quad (44)$$