Final Exam (70%)
SCE3006 Advanced Control with Implementation
Friday 21. December 2012
kl. 9.00 - 12.00
SKIP THIS FRONT PAGE AND REPLACE WITH STANDARD FROM WORD FILE

David Di Ruscio

January 28, 2013

Task 1 (15%) (Continuous time optimal control)

Given an LQ optimal control objective defined over a finite time horizon, i.e.,

$$J = \frac{1}{2}x(t_1)^T S x(t_1) + \frac{1}{2}\int_{t_0}^{t_1} [y^T Q y + u^T P u] dt, \tag{1}$$

where $S \in \mathbb{R}^{n\times n}$, $Q \in \mathbb{R}^{m\times m}$ and $P \in \mathbb{R}^{r\times r}$ are symmetric weighting matrices.

a) Consider a given discrete state space model

$$\dot{x} = Ax + Bu, \tag{2}$$

$$y = Dx, \tag{3}$$

for a process. Find the optimal control which minimize the objective, J, given by (1) subject to the model (2) and (3).

The solution should consist of:

1. An expression for the optimal control vector, u^*.
2. A Riccati equation.
3. A final value condition for the Riccati equation.

b) Consider now an infinite horizon LQ control objective as follows

$$J_i = \frac{1}{2}\int_{0}^{\infty} [x^T Q x + u^T P u] dt, \tag{4}$$

and the model as in (2) and (3) above.

What is now the solution to the LQ optimal control problem?

c) What is the requirements for the closed loop system to be stable ? Tips: The answer involves all matrices A, B, D, P, Q.

Task 2 (10%) (Discrete time optimal control)

Given an LQ optimal criterion defined over the time interval $0 \leq k \leq \infty$, i.e.,

$$J_0 = \frac{1}{2}\sum_{k=0}^{\infty}[x_k^T Q x_k + u_k^T P u_k], \tag{5}$$

where $Q \in \mathbb{R}^{m\times m}$ and $P \in \mathbb{R}^{r\times r}$ are symmetric weighting matrices.

a) Assume that the system is modeled with a state space model, i.e.,

$$x_{k+1} = Ax_k + Bu_k, \tag{6}$$

Find the optimal controller which are minimizing the objective J_0 given by (5) and subject to the model (6).

The solution should consist of:

1. An expression for the optimal control u_k^*.
2. A discrete Riccati equation.

b) Assume that the system is given by the scalar system

$$x_{k+1} = ax_k + bu_k, \tag{7}$$

and given the objective

$$J_0 = \frac{1}{2}\sum_{k=0}^{\infty}[qx_k^2 + pu_k^2]. \tag{8}$$

Find an analytic solution to the LQ optimal control problem.

Hint: You may use the results from Task 2a) above.

Task 3 (30%) (Discrete time LQ optimal control with Integral Action)

We are in this task to study an LQ optimal controller for a system described by the state space model

$$x_{k+1} = Ax_k + Bu_k + v, \tag{9}$$

$$y_k = Dx_k + w, \tag{10}$$

where v and w are constant and unknown disturbances.
Subject to the above state space model we want to design an LQ optimal controller which minimizes the following LQ criterion

$$J_i = \frac{1}{2}\sum_{k=i}^{\infty}((r - y_k)^T Q(r - y_k) + \Delta u_k^T P \Delta u_k). \tag{11}$$

where $\Delta u_k = u_k - u_{k-1}$ and r is a constant reference vector. Q and P are symmetric and positive semidefinite matrices.

a) Show that it is possible to write the model in (9) and (10) on deviation form, i.e.,

$$\Delta x_{k+1} = A\Delta x_k + B\Delta u_k, \tag{12}$$
$$\Delta y_k = D\Delta x_k, \tag{13}$$

where

$$\Delta x_k = x_k - x_{k-1}, \ \ \Delta u_k = u_k - u_{k-1}, \ \ \Delta y_k = y_k - y_{k-1}. \tag{14}$$

What can be gained by doing this?

b) Show that the model in (12) and (13) can be written as follows

$$\tilde{x}_{k+1} = \tilde{A}\tilde{x}_k + \tilde{B}\Delta u_k, \tag{15}$$
$$\tilde{y}_k = \tilde{D}\tilde{x}_k, \tag{16}$$

where

$$\tilde{x}_k = \begin{bmatrix} \Delta x_k \\ r - y_{k-1} \end{bmatrix}, \ \ \tilde{y}_k = r - y_k. \tag{17}$$

Here you should define the matrices $\tilde{A}$, $\tilde{B}$ and $\tilde{D}$.

c) The state space model in 2b) and the LQ criterion in (11) defines a standard discrete LQ optimal control problem of the form

$$\tilde{x}_{k+1} = \tilde{A}\tilde{x}_k + \tilde{B}\tilde{u}_k, \tag{18}$$
$$\tilde{y}_k = \tilde{D}\tilde{x}_k, \tag{19}$$

with LQ criterion

$$J_i = \frac{1}{2}\sum_{k=i}^{\infty}(\tilde{y}_k^T Q\tilde{y}_k + \tilde{u}_k^T P\tilde{u}_k), \tag{20}$$

where we for simplicity has defined

$$\tilde{u}_k = \Delta u_k. \tag{21}$$

Write down the solution to the LQ optimal control problem, of the form

$$\tilde{u}_k^* = \tilde{G}\tilde{x}_k. \tag{22}$$

the solution should consist of:

1. A discrete Riccati equation
2. an expression for the controller matrix $\tilde{G}$.
3. Write down an expression for the actual control of the form

$$u_k = f(\cdot) \tag{23}$$

which are to be used in order to control the process.

Task 4 (15%) (Receding horizon optimal control)

Given a system described by the continuous linear state space model

$$\dot{x} = Ax + Bu. \tag{24}$$

We want to study the LQ optimal control problem as described in the following. Consider the LQ optimal control objective defined over a receding horizon from present time t to a future time instant $t+T$, i.e.,

$$J = \frac{1}{2} x(t+T)^T S x(t+T) + \frac{1}{2} \int_t^{t+T} [x^T Q x + u^T P u] dt, \tag{25}$$

where $T > 0$ is a constant prediction horizon, $S \in \mathbb{R}^{n \times n}$, $Q \in \mathbb{R}^{m \times m}$ and $P \in \mathbb{R}^{r \times r}$ are as usual given symmetric weighting matrices.
From the maximum principle and $\dot{x} = \frac{\partial H}{\partial p}$, $\frac{\partial H}{\partial u} = 0$ and $\dot{p} = -\frac{\partial H}{\partial x}$ where H is the Hamiltonian function, we may deduce the autonomous system

$$\dot{\tilde{x}} = F\tilde{x} \tag{26}$$

where

$$\tilde{x} = \begin{bmatrix} x \\ p \end{bmatrix}, \quad \dot{\tilde{x}} = \begin{bmatrix} \dot{x} \\ \dot{p} \end{bmatrix}. \tag{27}$$

a) Find the system matrix F.

b) Use that $p(t+T) = Sx(t+T)$ and find R in the linear relationship $p = Rx$.

Hints: You should in this task use that the solution to the autonomous system (26) is given by

$$\tilde{x}(t+T) = e^{FT}\tilde{x}(t) = \Phi\tilde{x}(t) \tag{28}$$

where we also can partition the transition matrix $\Phi = e^{FT}$ as

$$e^{FT} = \Phi = \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{bmatrix}. \tag{29}$$

R is a function of the sub-matrices in Φ and S.

Remarks : R is the solution to the Riccati equation of this problem. Furthermore R is constant since the transition matrix Φ is constant.

1 Solution

$$\begin{pmatrix} \frac{\sqrt{(p\,a^2 - p\,a\,2 + q\,b^2 + p)\,(p\,a^2 + p\,a\,2 + q\,b^2 + p)} - p + a^2\,p + b^2\,q}{2\,b^2} \\ -\frac{p + \sqrt{(p\,a^2 - p\,a\,2 + q\,b^2 + p)\,(p\,a^2 + p\,a\,2 + q\,b^2 + p)} - a^2\,p - b^2\,q}{2\,b^2} \end{pmatrix} \tag{30}$$

Appendix

Continuous time optimal control

$$\dot{x} = f(x, u, t) \tag{31}$$

$$J = S(x(t_1)) + \int_{t_0}^{t_1} L(x, u, t)dt \tag{32}$$

The continuous time Maximum Principle

$$H = L + p^T f \tag{33}$$

$$\dot{p} = -\frac{\partial H}{\partial x} \tag{34}$$

$$p(t_1) = \frac{\partial S}{\partial x}|_{t_1} \tag{35}$$

Discrete time optimal control

$$x_{k+1} - x_k = f(x_k, u_k, k) \tag{36}$$

$$J_i = S(x_N) + \sum_{k=i}^{N-1} L(x_k, u_k, k) \tag{37}$$

The discrete time maximum Principle

$$H_k = L(x_k, u_k, k) + p_{k+1}^T (x_{k+1} - x_k) \tag{38}$$

$$p_{k+1} - p_k = -\frac{\partial H_k}{\partial x_k} \tag{39}$$

$$p_N = \frac{\partial S}{\partial x_N} \tag{40}$$

Derivation rules

$$\frac{\partial}{\partial x}(Qx) = Q^T, \quad \frac{\partial}{\partial x}(x^T Q x) = Qx + Q^T x \tag{41}$$

$$\frac{\partial}{\partial x}((r - Dx)^T Q(r - Dx)) = -2D^T Q(r - Dx) \tag{42}$$