

Final Exam (70%)
SCE3006 Advanced Control with
Implementation
Monday 17. December 2010
kl. 9.00 - 12.00

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December 17, 2010

Task 1 (20%) (Continuous time optimal control)

Given an LQ optimal control objective defined over a finite time horizon, i.e.,

$$J = \frac{1}{2}x(t_1)^T Sx(t_1) + \frac{1}{2} \int_{t_0}^{t_1} [y^T Qy + u^T Pu]dt, \quad (1)$$

where $S \in \mathbb{R}^{n \times n}$, $Q \in \mathbb{R}^{m \times m}$ and $P \in \mathbb{R}^{r \times r}$ are symmetric weighting matrices.

a) Consider a given discrete state space model

$$\dot{x} = Ax + Bu, \quad (2)$$

$$y = Dx, \quad (3)$$

for a process. Find the optimal control which minimize the objective, J , given by (1) subject to the model (2) and (3).

The solution should consist of:

1. An expression for the optimal control vector, u^* .
2. A Riccati equation.
3. A final value condition for the Riccati equation.

b) Consider now an infinite horizon LQ control objective as follows

$$J_i = \frac{1}{2} \int_0^\infty [x^T Qx + u^T Pu]dt, \quad (4)$$

and the model as in (2) and (3) above.

What is now the solution to the LQ optimal control problem?

c) Consider now an LQ control objective

$$J_i = \frac{1}{2} \int_t^{t+L} [x^T Qx + u^T Pu]dt, \quad (5)$$

where t is present time and L a constant prediction horizon.

What is now the solution to the LQ optimal control problem?

Task 2 (10%) (Discrete time optimal control)

Given an LQ optimal criterion defined over the time interval $0 \leq k \leq \infty$, i.e.,

$$J_0 = \frac{1}{2} \sum_{k=0}^{\infty} [y_k^T Qy_k + u_k^T Pu_k], \quad (6)$$

where $Q \in \mathbb{R}^{m \times m}$ and $P \in \mathbb{R}^{r \times r}$ are symmetric weighting matrices.

- a) Assume now that the system is modeled with a state space model with a direct feed-through in the output equation, i.e.,

$$x_{k+1} = Ax_k + Bu_k, \quad (7)$$

$$y_k = Dx_k + Eu_k. \quad (8)$$

Find the optimal controller which are minimizing the objective J_0 given by (6) and subject to the model (7) and (8)

The solution should consist of:

1. An expression for the optimal control u_k^* .
2. A discrete Riccati equation.

Task 3 (30%) (Discrete time LQ optimal control with Integral Action)

We are in this task to study an LQ optimal controller for a system described by the state space model

$$x_{k+1} = Ax_k + Bu_k + v, \quad (9)$$

$$y_k = Dx_k + w, \quad (10)$$

where v and w are constant and unknown disturbances.

Subject to the above state space model we want to design an LQ optimal controller which minimizes the following LQ criterion

$$J_i = \frac{1}{2} \sum_{k=i}^{\infty} ((r - y_k)^T Q (r - y_k) + \Delta u_k^T P \Delta u_k). \quad (11)$$

where $\Delta u_k = u_k - u_{k-1}$ and r is a constant reference vector. Q and P are symmetric and positive semidefinite matrices.

- a) Show that it is possible to write the model in (9) and (10) on deviation form, i.e.,

$$\Delta x_{k+1} = A\Delta x_k + B\Delta u_k, \quad (12)$$

$$\Delta y_k = D\Delta x_k, \quad (13)$$

where

$$\Delta x_k = x_k - x_{k-1}, \quad \Delta u_k = u_k - u_{k-1}, \quad \Delta y_k = y_k - y_{k-1}. \quad (14)$$

What can be gained by doing this?

b) Show that the model in (12) and (13) can be written as follows

$$\tilde{x}_{k+1} = \tilde{A}\tilde{x}_k + \tilde{B}\Delta u_k, \quad (15)$$

$$\tilde{y}_k = \tilde{D}\tilde{x}_k, \quad (16)$$

where

$$\tilde{x}_k = \begin{bmatrix} \Delta x_k \\ r - y_{k-1} \end{bmatrix}, \quad \tilde{y}_k = r - y_k. \quad (17)$$

Here you should define the matrices \tilde{A} , \tilde{B} and \tilde{D} .

c) The state space model in 2b) and the LQ criterion in (11) defines a standard discrete LQ optimal control problem of the form

$$\tilde{x}_{k+1} = \tilde{A}\tilde{x}_k + \tilde{B}\tilde{u}_k, \quad (18)$$

$$\tilde{y}_k = \tilde{D}\tilde{x}_k, \quad (19)$$

with LQ criterion

$$J_i = \frac{1}{2} \sum_{k=i}^{\infty} (\tilde{y}_k^T Q \tilde{y}_k + \tilde{u}_k^T P \tilde{u}_k), \quad (20)$$

where we for simplicity has defined

$$\tilde{u}_k = \Delta u_k. \quad (21)$$

Write down the solution to the LQ optimal control problem, of the form

$$\tilde{u}_k^* = \tilde{G}\tilde{x}_k. \quad (22)$$

the solution should consist of:

1. A discrete Riccati equation
2. an expression for the controller matrix \tilde{G} .
3. Write down an expression for the actual control of the form

$$u_k = f(\cdot) \quad (23)$$

which are to be used in order to control the process.

Task 4 (10%) (Diverse questions)

- a) What is the meaning of the "duality principle" ? write down a table which illustrates the "duality principle" for continuous systems.
- b) What is an LQG controller? Sketch a block diagram for a system controlled by an LQG controller

Appendix

Continuous time optimal control

$$\dot{x} = f(x, u, t) \quad (24)$$

$$J = S(x(t_1)) + \int_{t_0}^{t_1} L(x, u, t) dt \quad (25)$$

The continuous time Maximum Principle

$$H = L + p^T f \quad (26)$$

$$\dot{p} = -\frac{\partial H}{\partial x} \quad (27)$$

$$p(t_1) = \frac{\partial S}{\partial x} \Big|_{t_1} \quad (28)$$

Discrete time optimal control

$$x_{k+1} - x_k = f(x_k, u_k, k) \quad (29)$$

$$J_i = S(x_N) + \sum_{k=i}^{N-1} L(x_k, u_k, k) \quad (30)$$

The discrete time maximum Principle

$$H_k = L(x_k, u_k, k) + p_{k+1}^T (x_{k+1} - x_k) \quad (31)$$

$$p_{k+1} - p_k = -\frac{\partial H_k}{\partial x_k} \quad (32)$$

$$p_N = \frac{\partial S}{\partial x_N} \quad (33)$$

Derivation rules

$$\frac{\partial}{\partial x}(Qx) = Q^T, \quad \frac{\partial}{\partial x}(x^T Qx) = Qx + Q^T x \quad (34)$$

$$\frac{\partial}{\partial x}((r - Dx)^T Q(r - Dx)) = -2D^T Q(r - Dx) \quad (35)$$