Sluttprøve i fag A3802
Avansert reguleringsteknikk med programmering
mandag 11. desember 2006
kl. 9.00 - 12.00

Sluttprøven består av: 3 oppgaver.
Oppgaven teller 70 % av sluttkarakteren.
Det er 4 sider i sluttprøven.
Tillatte hjelpemidler: vedlegg til oppgaven

Faglig kontakt under eksamen:
Navn: David Di Ruscio
Tlf: 51 68, Rom: B249

Kybernetikk og industriell IT
Institutt for elektro, IT og kybernetikk
Avdeling for teknologiske fag
Høgskolen i Telemark
N-3914 Porsgrunn

Task 1 (32%) (discrete LQ optimal control)

Given an LQ criterion defined over the time horizon $i \leq k \leq N$, i.e.,

$$J_i = \frac{1}{2}x_N^T S x_N + \frac{1}{2}\sum_{k=i}^{N-1}[y_k^T Q y_k + u_k^T P u_k], \tag{1}$$

where $S \in \mathbb{R}^{n\times n}$, $Q \in \mathbb{R}^{m\times m}$ and $P \in \mathbb{R}^{r\times r}$ are symmetric weighting matrices.

a) Consider given a discrete time linear state space model

$$x_{k+1} = Ax_k + Bu_k, \tag{2}$$
$$y_k = Dx_k. \tag{3}$$

Find the optimal control which minimizes the criterion J_i given by Equation (1), subject to the model given by equations (2) and (3).

The solution should consist of:

1. An expression for the optimal control, u_k^*.
2. A discrete time Riccati equation.
3. Boundary condition for the Riccati equation.

b) Consider now the LQ criterion

$$J_i = \frac{1}{2}\sum_{k=i}^{\infty}(x_k^T Q x_k + 2x_k^T N u_k + u_k^T P u_k], \tag{4}$$

where $N \in \mathbb{R}^{n\times r}$, $Q \in \mathbb{R}^{m\times m}$ and $P \in \mathbb{R}^{r\times r}$ are weighting matrices.

Find the optimal control which minimizes the criterion J_i given by Equation (4), subject to the model given by equations (2) and (3).

The solution should consist of:

1. An expression for the optimal control, u_k^*.
2. A discrete time Riccati equation.
3. What about Boundary condition for the Riccati equation in this case.

c) Consider now given a discrete time linear state space model

$$x_{k+1} = Ax_k + Bu_k, \tag{5}$$
$$y_k = Dx_k + Eu_k, \tag{6}$$

and an LQ criterion

$$J_i = \frac{1}{2}\sum_{k=i}^{\infty}(y_k^T Q y_k + u_k^T P u_k], \tag{7}$$

1. Show that the LQ criterion (7) with the output Equation (6) can be written as the criterion in (4).
2. Commenting upon the solution to the LQ optimal control problem in this case.

Task 2 (6%) (Diverse questions)

a)

Given a scalar system

$$x_{k+1} = ax_k + bu_k, \tag{8}$$

and the LQ criterion

$$J_0 = \frac{1}{2}\sum_{k=0}^{\infty}(qx_k^2 + pu_k^2). \tag{9}$$

Find an analytically expression for the solution to this LQ optimal control problem, i.e., find an expression for the optimal control, u_k^*, which minimizes the criterion (9) subject to the state equation (8).

b) What is an LQG controller (only give a short description!).

Task 3 (32%)
(Discrete LQ optimal control with Integral Action)

We are in this task to study an LQ optimal controller for a system described by the state space model

$$\begin{aligned} x_{k+1} &= Ax_k + Bu_k + v, & (10)\\ y_k &= Dx_k + w, & (11) \end{aligned}$$

where v and w are constant and unknown disturbances.
Subject to the above state space model we want to design an LQ optimal controller which minimizes the following LQ criterion

$$J_i = \frac{1}{2}\sum_{k=i}^{\infty}((r-y_k)^T Q(r-y_k) + \Delta u_k^T P \Delta u_k). \quad (12)$$

where $\Delta u_k = u_k - u_{k-1}$ and r is a constant reference vector. Q and P are symmetric and positive semidefinite matrices.

a) Show that it is possible to write the model in (10) and (11) on deviation form, i.e.,

$$\begin{aligned} \Delta x_{k+1} &= A\Delta x_k + B\Delta u_k, & (13)\\ \Delta y_k &= D\Delta x_k, & (14) \end{aligned}$$

where

$$\Delta x_k = x_k - x_{k-1},\ \ \Delta u_k = u_k - u_{k-1},\ \ \Delta y_k = y_k - y_{k-1}. \quad (15)$$

What can be gained by doing this?

b) Show that the model in (13) and (14) can be written as follows

$$\begin{aligned} \tilde{x}_{k+1} &= \tilde{A}\tilde{x}_k + \tilde{B}\Delta u_k, & (16)\\ \tilde{y}_k &= \tilde{D}\tilde{x}_k, & (17) \end{aligned}$$

where

$$\tilde{x}_k = \begin{bmatrix} \Delta x_k \\ r - y_{k-1} \end{bmatrix},\ \ \tilde{y}_k = r - y_k. \quad (18)$$

Here you should define the matrices $\tilde{A}$, $\tilde{B}$ and $\tilde{D}$.

c) The state space model in 2b) and the LQ criterion in (12) defines a standard discrete LQ optimal control problem of the form

$$\begin{aligned} \tilde{x}_{k+1} &= \tilde{A}\tilde{x}_k + \tilde{B}\tilde{u}_k, && (19) \\ \tilde{y}_k &= \tilde{D}\tilde{x}_k, && (20) \end{aligned}$$

with LQ criterion

$$J_i = \frac{1}{2}\sum_{k=i}^{\infty}(\tilde{y}_k^T Q \tilde{y}_k + \tilde{u}_k^T P \tilde{u}_k), \tag{21}$$

where we for simplicity has defined

$$\tilde{u}_k = \Delta u_k. \tag{22}$$

Write down the solution to the LQ optimal control problem, of the form

$$\tilde{u}_k = \tilde{G}\tilde{x}_k. \tag{23}$$

the solution should consist of:

1. a discrete Riccati equation
2. an expression for the controller matrix $\tilde{G}$.
3. Write down an expression for the actual control of the form

$$u_k = f(\cdot) \tag{24}$$

which are to be used in order to control the process.

Tips: in Step 1 and 2 above you may use the results from task 1.

d) A PI controller may in the Laplace plane be written as follows

$$u = K_p \frac{1 + T_i s}{T_i s}(r - y). \tag{25}$$

1. Write down an continuous time state space model of the PI controller in (25).
2. Write down a discrete time state space model of the PI controller. You may use the explicit Euler method for discretization.
3. Write the discrete PI controller on deviation form and compare it with the LQ optimal controller in Step 2c).

Appendix

Continuous time optimal control

$$\begin{aligned} \dot{x} &= f(x,u,t) && (26) \\ J &= S(x(t_1)) + \int_{t_0}^{t_1} L(x,u,t)dt && (27) \end{aligned}$$

The continuous time Maximum Principle

$$\begin{aligned} H &= L + p^T f && (28) \\ \dot{p} &= -\frac{\partial H}{\partial x} && (29) \\ p(t_1) &= \frac{\partial S}{\partial x}|_{t_1} && (30) \end{aligned}$$

Discrete time optimal control

$$\begin{aligned} x_{k+1} - x_k &= f(x_k, u_k, k) && (31) \\ J_i &= S(x_N) + \sum_{k=i}^{N-1} L(x_k, u_k, k) && (32) \end{aligned}$$

The discrete time maximum Principle

$$\begin{aligned} H_k &= L(x_k, u_k, k) + p_{k+1}^T (x_{k+1} - x_k) && (33) \\ p_{k+1} - p_k &= -\frac{\partial H_k}{\partial x_k} && (34) \\ p_N &= \frac{\partial S}{\partial x_N} && (35) \end{aligned}$$

Derivation rules

$$\frac{\partial}{\partial x}(Qx) = Q^T, \quad \frac{\partial}{\partial x}(x^T Q x) = Qx + Q^T x \qquad (36)$$

$$\frac{\partial}{\partial x}((r - Dx)^T Q (r - Dx)) = -2D^T Q (r - Dx) \qquad (37)$$