Task 3, Exam 2023

$N=10, L=2, J=2, g=0$

a)

$$Y_{J+1|L} = Y_{3|2} = \overbrace{\begin{bmatrix} y_3 & y_4 & y_5 & y_6 & y_7 & y_8 \\ y_4 & y_5 & y_6 & y_7 & y_8 & y_9 \end{bmatrix}}^{K=6 \text{ columns}} \in \mathbb{R}^{2\times 6}$$

$N-1$

$$Y_{J|L} = Y_{2|2} = \begin{bmatrix} y_2 & y_3 & y_4 & y_5 & y_6 & y_7 \\ y_3 & y_4 & y_5 & y_6 & y_7 & y_8 \end{bmatrix} \in \mathbb{R}^{2\times 6}$$

$$U_{J|L+g} = U_{2|2} = \begin{bmatrix} u_2 & & u_7 \\ u_3 & \dots & u_8 \end{bmatrix} \in \mathbb{R}^{2\times 6}$$

$$U_{J|L+g-1} = U_{2|1} = [u_2 \; u_3 \; u_4 \; u_5 \; u_6 \; u_7] \in \mathbb{R}^{1\times 6}$$

$$O_L = O_2 = \begin{bmatrix} D \\ DA \end{bmatrix}, \quad H_L^d = H_2^d = \begin{bmatrix} E \\ DB \end{bmatrix}$$

$E=0$

$$\tilde{A}_L = O_L A (O_L^T O_L)^{-1} O_L^T, \quad \tilde{B}_L = [O_L B \;\; H_L^d] - \tilde{A}_L [H_L^d \;\; 0]$$

b) Remove noise ~~eu~~ ~~ilt~~

$$W = \begin{bmatrix} U_{0|J} \\ Y_{0|J} \\ U_{J|L+g} \end{bmatrix} \qquad Y_{J|L}/W = H_L^d U_{J|L+g-1}/W + O_L X_J/W$$

- General system

$$Z_{J+1|L} = (Y_{J+1|L}/W) U_{J|L+g}^{\perp}, \quad Z_{J|L} = (Y_{J|L}/W) U_{J|L+g}^{\perp}$$

- Deterministic case, $e_k = 0$, no noise

$$x_{k+1} = A x_k + B u_k$$
$$y_k = D x_k + E u_k$$

Eqs. (14) and (15) becomes, may put $J = 0$

$$Y_{0|L} = O_L X_0 + H_L^d U_{0|L+g-1} \qquad (1)$$

$$Y_{1|L} = \tilde{A}_L Y_{0|L} + \tilde{B}_L U_{0|L+g} \qquad (2)$$

May remove ~~noise~~ deterministic terms in (1) and (2) with

$$U_{0|L+g}^{\perp} = I - U_{0|L+g}^T (U_{0|L+g} U_{0|L+g}^T)^{+} U_{0|L+g}$$

$$Z_{J|L} = Y_{0|L} U_{0|L+g}^{\perp} = O_L \overbrace{X_0 U_{0|L+g}^{\perp}}^{\tilde{X}_0} = O_L \tilde{X}_0$$

$$Z_{J+1|L} = Y_{J|L} U_{0|L+g}^{\perp} \quad \text{and} \quad Z_{J+1|L} = \tilde{A}_L Z_{0|L}$$

c) Take SVD of $Z_{J|L}$

$$U S V^T = \text{svd}(Z_{J|L}) \approx U_1 S_1 V_1^T$$

and Then $Z_{J|L} = U_1 S_1 V_1^T = O_L \tilde{X}_0$

n = Number of non-~~zero~~ zero singular values, $S_1 \in \mathbb{R}^{n \times n}$

$$O_L = U_1$$

$$O_L = \begin{bmatrix} D \\ DA \\ \vdots \end{bmatrix} \longrightarrow D \text{ first block row in } U_1 = O_L$$

- Solve $\underbrace{Z_{J+1|L} = \overbrace{O_L A (O_L^T O_L)^{-1} O_L^T}^{\tilde{A}_L} \cdot \overbrace{U_1 S_1 V_1^T}^{Z_{J|L}}}$ for A

$$A = U_1^T Z_{J+1|L} V_1 S_1^{-1}$$

$Z_{J+1|L} = U_1 A S_1 V_1^T$, $O_L = U_1$, $U_1^T U_1 = I$

d) Comparing models (11)+(12) and (18)+(19) shows that

$$\varepsilon_k = F e_k \Rightarrow e_k = F^{-1}\varepsilon_k \qquad (3)$$

Putting (3) in (11) gives

$$C e_k = C F^{-1} \varepsilon_k \qquad K = CF^{-1}$$

$$x_{k+1} = A x_k + B u_k + C F^{-1} \varepsilon_k = A x_k + B u_k + K \varepsilon_k$$

e) Here we identify the noise term ε_k in model

$$x_{k+1} = A x_k + [B \; K]\begin{bmatrix} u_k \\ \varepsilon_k \end{bmatrix}$$

$$y_k = D x + [E \; I]\begin{bmatrix} u_k \\ \varepsilon_k \end{bmatrix}$$

Hence, the general system is reduced to a deterministic system

$$\varepsilon_k = Y_{J|1} - Y_{J|1} / \begin{bmatrix} U_{0|J} \\ Y_{0|J} \end{bmatrix}, \quad k \geq J$$