Kalman filter aposteriori state estimate

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Abstract

A proof of the formula $\hat{x}_k = \bar{x}_k + K(y_k - D\bar{x}_k)$ for the aposteriori state estimate $\hat{x}_k$ of the state vector x_k are given, assuming given a known observation/mesurement y_k and an apriori state estimate $\bar{x}_k$ of x_k.

1 Minimum variance aposteriori state estimate

We will assume known measurements y_k given by the linear equation

$$y_k = Dx_k + w_k. \tag{1}$$

Furthermore we assume that we have some knowledge of the true state x_k, i.e. an apriori state estimate $\bar{x}_k$.

Given a performance index as the weighted sum

$$J = \frac{1}{2}((x_k - \bar{x}_k)^T P(x_k - \bar{x}_k) + (y_k - Dx_k)^T Q(y_k - Dx_k)). \tag{2}$$

The scalar performance index J in Eq. (2) is a weighted quadratic form/sum of the error $x_k - \bar{x}_k$ between the true state vector x_k and the apriori known state estimate $\bar{x}_k$ and the error $y_k - Dx_k$. The problem is to find a better aposteriori state estimate $\hat{x}_k$ of the state vector x_k.

Using the model Eq. (1) shows that the error is the measurements noise $w_k = y_k - Dx_k$. From this it may be shown that the optimal weights are given by

$$P = \bar{X}_k^{-1},\ Q = W^{-1}, \tag{3}$$

where $E(e_k e_k^T) = W$ and $\bar{X} = E((x_k - \bar{x}_k)((x_k - \bar{x}_k)^T)$.

An optimal aposteriori state estimate $\hat{x}_k$ is found as the minimum of the performance index J. Hence we solve $\frac{dJ}{dx} = 0$ for the state x_k and putting the solution rqual to the aposteriori state estimate, $\hat{x}_k = x_k$. Using the Kernal rule with $J' = \frac{1}{2}u^T Q u$ and $u = y_k - Dx_k$

$$\frac{dJ}{dx} = P(x_k - \bar{x}_k) + \frac{du}{dx}\frac{dJ'}{du} = P(x_k - \bar{x}_k) - D^T Q(y_k - Dx_k) = 0. \tag{4}$$

This gives

$$(P + D^T Q D)x_k = P\bar{x}_k + D^T Q y_k, \tag{5}$$

and by adding $-D^T QD\bar{x}_k + D^T QD\bar{x}_k$ on the left hand side of Eq. (5) and rearranging, the famous result

$$\hat{x}_k = \bar{x}_k + K(y_k - D\bar{x}_k), \tag{6}$$

where

$$K = (P + D^T QD)^{-1} D^T Q = P^{-1} D^T (DP^{-1} D^T + Q^{-1})^{-1}. \tag{7}$$

The last equality in Eq. (7) is found by using the Matrix Inversion Lemma (MIL). However, it is quite simple to proof the equality in Eq. (7) so we skip the proof using the MIL. Equality in Eq. (7) is simply seen by pre multiplying with $(P + D^T QD)$ and post multiplying with $(DP^{-1} D^T + Q^{-1})$ in Eq. (7).

Using the expressions for the optimal weights in Eq. (3) we obtain the more used formulation of the Kalman filter gain matrix

$$K = \bar{X}_k D^T (W + D\bar{X}_k D^T)^{-1}. \tag{8}$$

References

This is from Balchen Stokastiske II lecture notes