Master study
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IIA2217 System Identification and Optimal Estimation

Exercise 1

Task 1

Introduction

many of the methods for design of control systems in courses as Control theory, Advanced control theory and Predictive control, are based on a process model. This exercise is an introduction to the field of System Identification, i.e. how we can identify a dynamic process model from observed data. There are also some realisation theory in the exercise. In optimal control theory we often use a discrete state space model of the following form:

$$x_{t+1} = Ax_t + Bu_t, \tag{1}$$
$$y_t = Dx_t + Eu_t, \tag{2}$$

for the computation of the optimal state feedback $u_t = G_t x_t$ and the feedback matrix G_t. Here x_t is an n dimensional state vector, u_t is an r dimensional control input vector and y_t is an m dimensional output ore measurement vector. The initial state vector at time $t = 0$ is given by x_0. The initial state may be estimated or computed if necessary.
This model may be obtained by e.g. first work out a physical model and thereafter linearize the model, if it is non linear. Another method of great interest is to identify the system order n and the state space model matrices (A, B, D, E) and the initial state x_0 directly from known observed input and output time series data, u_t and y_t. This means that the model is obtained by solving the following problem:
System Identification Problem
Given a number N of the process inputs u_t and the process outputs y_t, i.e. from known input and output data

$$\left.\begin{matrix} u_t \\ y_t \end{matrix}\right\} \quad \forall \quad t = 1, ..., N \tag{3}$$

compute the system order, n, and the state space model matrices (A, B, D, E).
Note that the sign $\forall$ are denoted *for al.* There are also common sense to organize the input and output data in data matrices U and Y such that

$$U = \begin{bmatrix} u_1^T \\ u_2^T \\ \vdots \\ u_N^T \end{bmatrix} \in \mathbb{R}^{N \times r}, \tag{4}$$

$$Y = \begin{bmatrix} y_1^T \\ y_2^T \\ \vdots \\ y_N^T \end{bmatrix} \in \mathbb{R}^{N \times m}. \tag{5}$$

We will later in the course show that the problem may be very effectively solved by use of the DSR subspace system identification method. The solution to this problem is of great practical interest because the method is simple and robust to use. The resulting model may be used for controller synthesis, optimal and predictive control, state estimation, prediction and so on. this exercise gives an introduction to some parts of the field of system identification.

Exercise tasks

1. Show that the output, y_t, from the discrete time model (1) and (2) can be written as

$$y_t = \overbrace{DA^t x_0}^{\text{Contribution from } x_0} + \overbrace{\sum_{i=1}^{t} DA^{t-i} B u_{i-1}}^{\text{Contribution from past inputs}} + E u_t \tag{6}$$

   The matrices $H_k = DA^{k-1}B$ for $k = 1, ..., t$ is called the impulse response matrices of the system. What is the dimension of the impulse response matrices?

2. Suppose that a Single Input and Single output (SISO) system i excited by a impulse $u_0 = 1$ at $t = 0$ and $u_t = 0$ for $t \geq 1$. The following input and output data are recorded.

$$u_0 = 1 \quad u_1 = 0 \quad u_2 = 0 \quad u_3 = 0 \quad u_4 = 0$$

$$y_0 = -1 \quad y_1 = 2 \quad y_2 = -1 \quad y_3 = -1.9 \quad y_4 = -1.93$$

   The time series data ia as usual stored as follows:

$$U = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \in \mathbb{R}^{5 \times 1}, \; Y = \begin{bmatrix} -1 \\ 2 \\ -1 \\ -1.9 \\ -1.93 \end{bmatrix} \in \mathbb{R}^{5 \times 1}. \tag{7}$$

   Find the impulse responses H_{k+1} for $k = 0, 1, 2, 3, 4$ for the system, i.e. estimate $H_1 = DB$, $H_2 = DAB$, $H_3 = DA^2B$ etc. Assume for simplicity that the initial state is zero, i.e. $x_0 = 0$.

3. Write down the expression for the observability matrix O_n for the matrix pair (D, A) and an expression for the controllability matrix C_n for the matrix pair (A, B). What is the dimensions for the matrices O_n and C_n ? The system order of the system is n. What is the rank of the matrices C_n and O_n? under the assumptions that the system is both observable and controllable.

4. Show that the following factorization of the Hankel matrix $\mathbf{H}_{1|L}$ is valid,

$$\mathbf{H}_{1|L} = O_L C_J \tag{8}$$

where $\mathbf{H}_{1|L}$ is a matrix of impulse response matrices given by

$$\mathbf{H}_{1|L} = \begin{bmatrix} DB & DAB & \cdots & DA^{J-1}B \\ DAB & DA^2B & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ DA^{L-1}B & DA^L B & \cdots & DA^{L+J-2}B \end{bmatrix} \tag{9}$$

The matrix $\mathbf{H}_{1|L}$ defined in Equation (9) is called a Hankel matrix because of the special structure of the matrix.

What is the rank of the extended controllability matrix C_J and the extended observability matrix O_L where we assume that $L \geq n - \text{rank}(D) + 1$ og $J \geq n - \text{rank}(B) + 1$? can you say something about the rank of the Hankel matrix $\mathbf{H}_{1|L}$?

5. It is possible to compute the impulse response matrices $DA^{k-1}B$ from known input and output time series data u_t and y_t. Hence, we can define the Hankel matrix $\mathbf{H}_{1|L}$ given by Equation (9). We will in the following give an introduction to realisation theory, i.e. the problem of computing the model matrices (A, B, D) from known Hankel matrices.

   Assume that the Hankel matrix $\mathbf{H}_{1|L}$ is known. Write down the Singular Value Decomposition (SVD) of the Hankel matrix $\mathbf{H}_{1|L}$. Compare this SVD with the factorization in Step 3 and find expressions for O_L and C_J as a function of the SVD of $\mathbf{H}_{1|L}$.

6. Assume that you have computed the extended observability matrix O_L and the extended controllability matrix C_J as in Step 4 above, ore that those matrices are known. How can the system matrices D and B directly be found from the known matrices O_L and C_J?

7. The system matrix A can be found from a similar technique. Show that the A matrix can be computed from a Hankel matrix $H_{2|L}$ and the known matrices O_L and C_J as computed above. Find a formula for computing A and find the numerical value for A.