Master study
EIK
Faculty of Technology, Natural Sciences and Maritime Sciences
University of South-Eastern Norway
DDiR, January 18, 2023

IIA2217 System Identification and Optimal Estimation

Solution proposal: Exercise 1

Task 1

1. The state space model consists of two equations, one state equation, $x_{t+1} = Ax_t + Bu_t$, and one output equation, $y_t = Dx_t + Eu_t$. We can now define the equations for the time instants $t = 0, 1, 2, 3, \ldots$ and so on. For simplicity and in order to illustrate we only use the four first time instants $t = 0, 1, 2, 3$. Hence we have

$$\left.\begin{array}{ll} t=0, & y_0 = Dx_0 + Eu_0, \\ & x_1 = Ax_0 + Bu_0. \\ t=1, & y_1 = Dx_1 + Eu_1 = DAx_0 + DBu_0 + Eu_1, \\ & x_2 = Ax_1 + Bu_1 = A^2x_0 + ABu_0 + Bu_1. \\ t=2, & y_2 = Dx_2 + Eu_2 = DA^2x_0 + DABu_0 + DBu_1 + Eu_2, \\ & x_3 = Ax_2 + Bu_2 = A^3x_0 + A^2Bu_0 + ABu_1 + Bu_2. \\ t=3, & y_3 = Dx_3 + Eu_3 = DA^3x_0 + DA^2Bu_0 + DABu_1 + DBu_2 + Eu_3. \end{array}\right\} \quad (1)$$

This gives

$$\left.\begin{array}{ll} t=0, & y_0 = Dx_0 + Eu_0, \\ t=1, & y_1 = DAx_0 + DBu_0 + Eu_1, \\ t=2, & y_2 = DA^2x_0 + DABu_0 + DBu_1 + Eu_2, \\ t=3, & y_3 = DA^3x_0 + DA^2Bu_0 + DABu_1 + DBu_2 + Eu_3. \end{array}\right\} \quad (2)$$

This can in general be written as

$$y_t = DA^t x_0 + \sum_{i=1}^{t} H_{t-i+1} u_{i-1} + Eu_t \quad (3)$$

where

$$H_{t-i+1} = DA^{t-i}B. \quad (4)$$

is the impulse response matrix for the system at time instant $t-i+1$. We have shown that the output from a linear discrete time state space model is equivalent to a impulse response model driven by the initial state x_0 and the inputs, u_t, of the system.

2. let us illustrate (2) by using the definition (3). Putting the definition (4) into (2) gives

$$\left.\begin{array}{ll} t=0, & y_0 = Dx_0 + Eu_0, \\ t=1, & y_1 = DAx_0 + H_1 u_0 + Eu_1, \\ t=2, & y_2 = DA^2 x_0 + H_2 u_0 + H_1 u_1 + Eu_2, \\ t=3, & y_3 = DA^3 x_0 + H_3 u_0 + H_2 u_1 + H_1 u_2 + Eu_3. \end{array}\right\} \tag{5}$$

We have given that $x_0 = 0$. This gives

$$\left.\begin{array}{ll} t=0, & y_0 = Eu_0, \\ t=1, & y_1 = H_1 u_0 + Eu_1, \\ t=2, & y_2 = H_2 u_0 + H_1 u_1 + Eu_2, \\ t=3, & y_3 = H_3 u_0 + H_2 u_1 + H_1 u_2 + Eu_3, \\ t=4, & y_4 = H_4 u_0 + H_3 u_1 + H_2 u_2 + H_1 u_3 + Eu_4. \end{array}\right\} \tag{6}$$

From the data matrix

$$U = \begin{bmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{7}$$

we have that

$$\left.\begin{array}{ll} t=0, & y_0 = Eu_0, \\ t=1, & y_1 = H_1 u_0, \\ t=2, & y_2 = H_2 u_0, \\ t=3, & y_3 = H_3 u_0, \\ t=4, & y_4 = H_4 u_0. \end{array}\right\} \tag{8}$$

Putting into numerical values gives the system parameter (matrix) E as follows

$$E = \frac{y_0}{u_0} = -1, \tag{9}$$

and the impulse responses

$$H_1 = \frac{y_1}{u_0} = 2, \quad H_2 = \frac{y_2}{u_0} = -1, \quad H_3 = \frac{y_3}{u_0} = -1.9, \quad H_4 = \frac{y_4}{u_0} = -1.93. \tag{10}$$

3. The controllability matrix C_n for the matrix pair (A, B) and its dimension are given by

$$C_n = \begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix} \in \mathbb{R}^{n \times nr}. \tag{11}$$

The observability matrix O_n for the matrix pair (D, A) and its dimension are given by

$$O_n = \begin{bmatrix} D \\ DA \\ DA^2 \\ \vdots \\ DA^{n-1} \end{bmatrix} \in \mathbb{R}^{mn \times n}. \tag{12}$$

If the system is controllable then we have that $\text{rank}(C_n) = n$ and if the system is observable then we have that $\text{rank}(O_n) = n$.

4. We define the extended controllability matrix C_J as follows

$$C_J = \begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B & \cdots & A^{J-1}B \end{bmatrix} \in \mathbb{R}^{n \times Jr}. \quad (13)$$

The matrix C_J is defined as an extended controllability matrix when $J > n$. In the same way an extended observability matrix O_L is defined as follows

$$O_L = \begin{bmatrix} D \\ DA \\ DA^2 \\ \vdots \\ DA^{L-1} \end{bmatrix} \in \mathbb{R}^{mL \times n}. \quad (14)$$

O_L is defined as an extended observability matrix when $L > n$.

From those definitions it is simple to prove Equations (8) and (9) in the Exercise 1 text, by simply multiply $O_L C_J$.

Furthermore we have that $\text{rank}(O_L) = n$ and $\text{rank}(C_J) = n$ when the system is both observable and controllable. Furthermore we can show that $\text{rank}(H_{1|L}) = \text{rank}(O_L C_J) = n$.

5. A Singular Value Decomposition (SVD) of the Hankel matrix $H_{1|L}$ is given by

$$H_{1|L} = USV^T = U_1 S_1 V_1^T + U_2 S_2 V_2^T \approx U_1 S_1 V_1^T \quad (15)$$

Comparing this with the relationship $H_{1|L} = O_L C_J$ shows that we can take

$$O_L = U_1, \quad C_J = S_1 V_1^T \quad (16)$$

This gives a so called output normal realization.

6. In Matlab notations we find D and B as follows

$$D = O_L(1:m,:) \quad (17)$$
$$B = C_J(:,1:r) \quad (18)$$

7. We can show that

$$H_{2|L} = O_L A C_J \quad (19)$$

From this we can compute A as follows

$$A = (O_L^T O_L)^{-1} O_L^T H_{2|L} C_J^T (C_J C_J^T)^{-1} \quad (20)$$

```
% Solution Exercise 1: Numerical part

U=[1 0 0 0 0]'; Y=[-1 2 -1 -1.9 -1.93]';

% Step 2) Impulse responses
H0=Y(1)/U(1);
H1=Y(2)/U(1);
H2=Y(3)/U(1);
H3=Y(4)/U(1);
H4=Y(5)/U(1);

% Step 5) Computations of O2 og C2
H12 =[H1 H2; H2 H3]

[U,S,V]=svd(H12);
% System order, n, equal to the number of singular values different from
% zero.
s=diag(S)
n=2
% Splitting up the SVD
S1=S(:,1:2); U1=U(:,1:2); V1=V(:,1:2);
O2=U1;                   % Observerbility matrix
C2=S1*V1';               % Controllability matrix
b=C2(:,1)
d=O2(1,:)

% Computing A
H22=[H2 H3; H3 H4]
a=pinv(O2'*O2)*O2'*H22*C2'*pinv(C2*C2')

% Test: Check if the model gives the impulse responses.
h1=d*b
h2=d*a*b
h3=d*a^2*b
h4=d*a^3*b
```