Master study
Systems and Control Engineering
Department of Technology
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Topic: System identification and optimal estimation

Exercise 6, State estimation and Kalman filter

Task 1

Given a Single Input and Single Output (SISO) system with one state described by the following model

$$\begin{aligned} \dot{x} &= ax + bu + cv, && (1) \\ y &= dx + w, && (2) \end{aligned}$$

where x is the internal state in the system, y is the measured output, v is an uncorrelated zero mean white noise process with given variance, w is an uncorrelated zero mean white noise process with given variance, i.e.,

$$E(v(t)) = 0 \quad \text{and} \quad E(v(t)v^T(t+\tau)) = q_0^2\delta(\tau) \tag{3}$$

$$E(w(t)) = 0 \quad \text{and} \quad E(w(t)w^T(t+\tau)) = r_0^2\delta(\tau) \tag{4}$$

where

$$\delta(\tau) = 1 \quad \text{for} \quad \tau = 0 \tag{5}$$

$$\delta(\tau) = 0 \quad \text{for} \quad \tau \neq 0 \tag{6}$$

Vi also define

$$\begin{aligned} V &= E(v(t)v^T(t)) = q_0^2, && (7) \\ W &= E(w(t)w^T(t)) = r_0^2, && (8) \end{aligned}$$

for further use in the exercise.

Remarks

q_0^2 is the variance (covariance) to the noise process v and q_0 is the standard deviation, i.e., the standard deviation is the square root of the variance. If v is a white noise vector then it have a covariance matrix $E(v(t)v^T(t)) = V$. This matrix is symmetric and positive definite, $V > 0$, when no noise variables in v are exactly identical to zero. Otherwise, the covariance matrix V will be positive semi definite, i.e. $V \geq 0$.

The Cholesky factorization of the covariance matrix can be computed for symmetric and positive definite matrices, e.g. when $V > 0$, i.e. $V = V_0V_0^T$ where V_0 is an upper triangular matrix. The Cholesky factorization is some thimes loosely spoken called a square root factorization. The standard deviation of

each noise variable, v_i, in the noise vector v is then given as the square root of the corresponding diagonal element in the covariance matrix V. The diagonal element v_{ii}, in the matrix V is the variance for the noise process v_i, where v_i is element number i in the vector v.
Note that if v is known at a (large) number N of discrete time instants then the covariance matrix V can be estimated (computed) as follows

$$V = \frac{1}{N} \sum_{t=0}^{N-1} v_t v_t^T \quad \textit{biased estimate} \tag{9}$$

$$V = \frac{1}{N-1} \sum_{t=0}^{N-1} v_t v_t^T \quad \textit{unbiased estimate} \tag{10}$$

A state estimator for a linear dynamic system $\dot{x} = Ax + Bu + Cv$, $y = Dx + w$ is given by

$$\begin{aligned} \dot{\hat{x}} &= A\hat{x} + Bu + K(y - \hat{y}), & (11) \\ \hat{y} &= D\hat{x}, & (12) \end{aligned}$$

where the optimal Kalman filter gain matrix, K, is given by

$$K = XD^T W^{-1}, \tag{13}$$

where the covariance matrix of the estimation error $X = E((x - \hat{x})(x - \hat{x})^T)$ is given as the positive definite solution to the following matrix Riccati equation

$$\dot{X} = AX + XA^T - XD^T W^{-1} DX + CVC^T, \tag{14}$$

where the initial covariance matrix $X(t_0)$ is specified or given.

a) *Find the optimal stationary state estimator (stationar Kalman filter), for the system in (1) and (2). Note that the stationary Kalman filter is obtained by putting $\dot{X} = 0$ in (14). It can also be shown that this is the optimal filter for time invariant systems, i.e. for systems in which the model matrices A, C, D and so on is constant and not dependent on time, t.*

b) *Discuss the solution as functions of the standard deviations q_0 and r_0.*

c) *Given numerical values for the system parameters, i.e. $a = -0.1054$, $b = 0.5268$, $c = 0.6322$, $d = 1$, as well as standard deviations for the noise processes, $q_0 = 0.1$ and $r_0 = 0.1$. We also assume a sampling time $\Delta t = 1$ $[s]$.*

   - *Find the time constant of the system.*
   - *Find the Kalman filter gain, K, by putting numerical values into the solution found in Step 1a).*

- *Find the Kalman filter gain , K, by using the MATLAB Control System Toolbox function* **lqe**.
- *Simulate the system with the state estimator in parallel. The simulations should be performed by using a for loop. Use the explicit Euler method in order to solve (discrete) the continuous equations. Discrete time white noise processes v_k and w_k can be obtained by using the MATLAB function* **randn**. *A sufficient simulation horizon may be five to ten times the time constant in the system, and also dependent on the variations in the input u.*

Task 2

a) *Make a discrete time state space model of the continuous model in Step 1c). use a zero order hold method for the discretization method, i.e. so that the variables u and v are constant over the sampling interval, $\Delta t = 1$. The model shoud be of the form*

$$x_{k+1} = ax_k + bu_k + cv_k, \tag{15}$$
$$y_k = dx_k + w_k. \tag{16}$$

Use the MATLAB function **c2dm** *or* **c2d** *in order to find the parameters a, b, c and d in the discrete time model.*

b) *Write down a Kalman filter on apriori aposteriori form. The Kalman filter gain can be computed by the MATLAB function* **dlqe**.

c) *Simulate the discrete time Kalman filter in Step b) above.*

d) *Write down the corresponding Kalman filter on innovations form.*

Task 3

Given a SISO one state system as described by the following model

$$\dot{x} = ax + bu + cv \tag{17}$$
$$y = x + w \tag{18}$$

The model is the same as in Task 1 but the noise process is not zero mean but it rather have a non-zero mean given by

$$E(v) = \bar{v} \tag{19}$$

a) *Assume that the noise is slowly varying. The noise can then be modelled as a so called random walk, i.e.*

$$v_{t+1} = v_t + \Delta t dv_t \quad \text{discrete noise model} \tag{20}$$
$$\dot{v} = dv \quad \text{continuous noise model} \tag{21}$$

where Δt is the sampling time. Note that the discrete noise model is obtained by discretizing the continuous model by using the explicit Euler method. dv is a zero mean white noise process with covariance q_0^2. Make an augmented model of the process model and the noise model which can be used to construct an optimal state estimator for both estimating x and the colored noise v.

b) Find an optimal estimator, kalman filter, for the augmented system. use an infinite time horizon, e.g. use the stationary Kalman filter equations. Use with advantage the MATLAB function lqe. Numerical values may be $a = -1$, $b = 0.5$, $c = 0.6$, $r_0 = 1$ and $0.01 \leq q_0 \leq 10$. Simulate the system by using a for loop as well as using the MATLAB function dlsim.