Master study
Systems and Control Engineering
Department of Technology
Telemark University College
DDiR, March 8, 2007

Topic: System identification and optimal estimation

Solution proposal: Exercise 6, state estimation and kalman filter

Task 1

a) The Riccati equation with $c = 1$ becomes

$$2aX - d^2W^{-1}X^2 + V = 0 \tag{1}$$

where we here are using $V = q_0$ and $W = r_0$. This gives the solution

$$X = \frac{ar_0}{d^2} + \sqrt{\frac{a^2r_0^2}{d^4} + \frac{q_0r_0}{d^2}} \tag{2}$$

The Kalman filter gain is then

$$K = XD^TW^{-1} = \frac{1}{d}(a + \sqrt{a^2 + d^2\frac{q_0}{r_0}}) \tag{3}$$

In case when $V = q_0^2$, $W = r_0^2$ and $c \neq 1$ as in the exercise text we have

$$K = XD^TW^{-1} = \frac{1}{d}(a + \sqrt{a^2 + (dc)^2\frac{V}{W}}) \tag{4}$$

The kalman filter is then given by

$$\dot{\hat{x}} = A\hat{x} + Bu + K(y - D\hat{x}), \tag{5}$$

$$\hat{y} = D\hat{x}. \tag{6}$$

Note that we need an initial value for the state estimate, $\hat{x}(t_0)$, in order to start the filter. This initial value can e.g. be specified from process knowledge, however, errors in the initial value will die out because of the fact thet the filter is stable.

b) The dynamics of the filter is given by the differential equation

$$\dot{\hat{x}} = (A - KD)\hat{x} + Bu + Ky, \tag{7}$$

We are using the above in Step 1a) and find that

$$\dot{\hat{x}} = -\sqrt{a^2 + d^2\frac{q_0}{r_0}}\hat{x} + bu + Ky. \tag{8}$$

As we see, the filter dynamics becomes faster when the ratio $\frac{q_0}{r_0}$ (or $\frac{q_0^2}{r_0^2}$) increases. It is this ratio which is the key in the filter dynamics. Note also that we in many circumstances can look at this ratio as a tuning factor in the filter, i.e. so that we can choose the ratio $\frac{q_0}{r_0}$ (or $\frac{q_0^2}{r_0^2}$) so that the estimate state $\hat{x}$ is sufficient. We are often putting $r_0 = 1$ and tuning q_0 in order to obtain a sufficient velocity in the state estimate.

Task 2 2

a) The Kalman filter on apriori aposteriori form for a linear time invariant system is given by

$$
\begin{aligned}
\bar{y}_k &= D\bar{x}_k && (9)\\
\hat{x}_k &= \bar{x}_k + K(y_k - \bar{y}_k && (10)\\
\bar{x}_{k+1} &= A\hat{x}_k + Bu_k && (11)
\end{aligned}
$$

We need an initial value for $\bar{x}_0$ in order to start up the estimator at time $k = 0$.

Task 3

Given the system

$$
\begin{aligned}
\dot{x} &= ax + bu + cv && (12)\\
y &= x + w && (13)
\end{aligned}
$$

The disturbance v is colored with non-zero mean. We are modelling v as an integrator exited by a zero mean white noise process dv, i.e.,

$$\dot{v} = dv \tag{14}$$

We then have the following augmented linear state space model

$$
\begin{aligned}
\begin{bmatrix} \dot{x} \\ \dot{v} \end{bmatrix} &= \begin{bmatrix} a & c \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ v \end{bmatrix} + \begin{bmatrix} b \\ 0 \end{bmatrix} u + \begin{bmatrix} 0 \\ 1 \end{bmatrix} dv && (15)\\
y &= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ v \end{bmatrix} + w && (16)
\end{aligned}
$$

We want to find the continuous kalman filter. the model is of the form

$$
\begin{aligned}
\dot{\tilde{x}} &= A\tilde{x} + Bu + Cdv && (17)\\
y &= D\tilde{x} + w && (18)
\end{aligned}
$$

The continuous and stationary Kalman filter is given by

$$
\begin{aligned}
AX + XA^T - XD^TW^{-1}DX + CVC^T = 0 && (19)\\
K = XD^TW^{-1} && (20)
\end{aligned}
$$

where W is the covariance of the measurements noise and V is the covariance of the process noise. We have

$$W = r_0^2 \quad (21)$$
$$V = q_0^2 = \mathrm{E}(dvdv^T) \quad (22)$$

Some computations gives

$$\begin{bmatrix} 2(ax_{11}+cx_{21}) & ax_{21}+cx_{22} \\ ax_{21}+cx_{22} & 0 \end{bmatrix} - W^{-1}\begin{bmatrix} x_{11}^2 & x_{11}x_{21} \\ x_{11}x_{21} & x_{21}^2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & V \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad (23)$$

Only two of the three above equations are necessary in order to compute x_{11} and x_{21}, i.e.,

$$2(ax_{11}+cx_{21}) - W^{-1}x_{11}^2 = 0 \quad (24)$$
$$-W^{-1}x_{21}^2 + V = 0 \quad (25)$$

This gives

$$x_{21} = \sqrt{VW} \quad (26)$$
$$x_{11} = \frac{a + \sqrt{(a^2 + 2cW^{-1}\sqrt{VW})}}{W^{-1}} \quad (27)$$

The Kalman filter gain matrix is given by

$$K = XD^TW^{-1} = \begin{bmatrix} k_{11} \\ k_{21} \end{bmatrix} = \begin{bmatrix} x_{11}W^{-1} \\ x_{21}W^{-1} \end{bmatrix} \quad (28)$$

This gives

$$k_{11} = a + \sqrt{(a^2 + 2c\sqrt{\frac{V}{W}})} \quad (29)$$
$$k_{21} = \sqrt{\frac{V}{W}} \quad (30)$$

We may also substitute for the variances V and W, i.e.,

$$k_{11} = a + \sqrt{(a^2 + 2c\frac{q_0}{r_0})} \quad (31)$$
$$k_{21} = \frac{q_0}{r_0} \quad (32)$$

As we see, it is the ratio between the standard deviation of the process noise, q_0, (or the variance $V = q_0^2$) and the standard deviation of the measurements noise, r_0 (or the variance $W = r_0^2$) which influences upon the elements in the Kalman filter gain K. As we see, k_{21}, increases when the ratio $\frac{q_0}{r_0}$ increases, i.e. a large process noise variance dives large gain, but large measurements noise variance gives small gain. Large measurements noise implies an uncertain measurement and the Kalman filter will relay more on the model, and therefore reduce the Kalman filter gain, K.
The Kalman filter is defined by the following equations

$$\dot{\hat{x}} = A\hat{x} + Bu + K(y - \hat{y}) \quad (33)$$
$$\hat{y} = D\hat{x} \quad (34)$$

We are guaranteed stability of the estimator when the pair (A, D) is observable and V and W are positive definite, and the assumptions for stability in LQ optimal systems are satisfied. The optimal estimator problem is dual to optimal LQ control. We can look at the eigenvalues of the estimator by analyzing

$$\dot{\hat{\tilde{x}}} = (A - KD)\hat{\tilde{x}} + Bu + Ky \tag{35}$$

$$\hat{y} = D\hat{\tilde{x}} \tag{36}$$

The filter or estimator can be looked upon as a separate system driven by measurements noise, y, and inputs u. The dynamics of this system is described by the eigenvalues of $A - KD$. Vw have

$$A - KD = \begin{bmatrix} f - k_{11} & c \\ -k_{21} & 0 \end{bmatrix} \tag{37}$$

we will here not go into further details of how the eigenvalues vary with V and W. But we will mention that we will obtain more degrees in freedom in order to control the dynamics of the filter if we instead describe the system as follows

$$\dot{x} = ax + bu + cv + dx \tag{38}$$

$$y = x + w \tag{39}$$

where dx is unknown zero mean white noise with variance V_x. dx may be interpreted as uncertainity resulting from e.g. un modelled effects. The augmented model is then influenced of both dx and dv. The augmented model will then have the covariance matrix

$$\tilde{V} = \begin{bmatrix} V_x & 0 \\ 0 & V \end{bmatrix} \tag{40}$$

Hence, we have obtained an additional degree of freedom which can be used to influence the estimator dynamics, namely the parameter V_x. We may now put (40) into Equation (23). The procedure for computing the Kalman filter gain is the same as that given above. It is important to note that we may obtain better state estimates if we are assuming that all state variable equations are influenced by noise.

Figure 1: Simulation results with variances $W = 1$ and $V = 1$. See also the MATLAB script file ov9s.m.

Figure 2: Simulation results with variances $W = 1$ and $V = 10$. See also the MATLAB script file ov9s.m.

```
% OV9S.M %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%  Formaal                                                                         %
%  Script for oving 9 oppgave 2                                                    %
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
a=-1; b=0.5; d=1; c=0.6;                % Prosessmodellen
W=1;                                    % Vekt"matrise" for maalestoeyen, W=E(w w')
V=10.0;                                 % Vekt "matrise" for stoeyen,     V=E(v v')
A=[a,c;0,0]; C=[0;1]; D=[1,0];          % Augmenterte systemmatriser
[l,x]=lqe(A,C,D,V,W);                   % Control system toolbox finksjon
k1=l(1,1);                              % Kalman forsterkning for estimering av x
k2=l(2,1);                              % Kalman forsterkning for estimering av v

at=[a,0,0;k1,a-k1,c;k2,-k2,0];          % Augmenter modell for simulering av
bt=[b;b;0];                             % system og estimator med feks. lsim.
ct=[c;0;0];
dt=eye(3);
et=zeros(3,2);

N=1200;                                 % Antall sampler
v=ones(N,1);                            % Simulerer med et sprang i forstyrrelsen
v(50:600,1)=v(50:600,1)+ones(551,1)*0.1;% Sprang ved sample 50
u=zeros(N,1);                           % Null paadrag, annet alternativ er randn(N,1)
t=0:N-1; t=t'/50;
x0=[0.6;0.6;1.0];                       % Initialverdier, stasjonaerverdier
y=lsim(at,[bt,ct],dt,et,[u,v],t,x0);    %Simulerer

subplot(211), plot(t,[y(:,1),y(:,2)])   % Plotter resultatene
grid
title('x1 og hat(x1): Tilstand og estimert tilstand')
subplot(212), plot(t,[y(:,3),v])
grid
title('v og hat(v): Forstyrrelsen og estimert forstyrrelse')
xlabel('[time (units)]')
```