Mandatory Exercise 4 2024
Solution Proposal

Tasks:

a) System Identification is to build mathematical models from observed data (of dynamic systems).

b) A Kalman filter is an optimal state observer. If $x_k \in \mathbb{R}^n$ is the true state vector and $\hat{x}_k$ the optimal kalman filter estimate, then the covariance matrix $\hat{X}$ of the error $x_k - \hat{x}_k$ is minimized ie.

$$\hat{X} = E\left((x_k - \hat{x}_k)(x_k - \hat{x}_k)^T\right) \text{ is minimized.}$$

Also

$$E\left((x_k - \hat{x}_k)(x_k - \hat{x}_k)^T\right) \approx \frac{1}{N} \sum_{k=1}^{N} (x_k - \hat{x}_k)(x_k - \hat{x}_k)^T$$

when N large.

c)
- $$\begin{cases} x_{k+1} = A x_k + B u_k \\ y_k = D x_k \end{cases}$$

- $H_k = D A^{k-1} B \quad \forall \; k = 1, 2, \ldots$

  is the impulse response matrix, $H_k \in \mathbb{R}^{m \times r}$.

d) Three equations and two unknowns,

$$1 = a+b, \quad 4 = 2a+b, \quad 5 = 3a+b$$

may be written in matrix form as

$$\begin{bmatrix} 1 \\ 4 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} \quad \text{or} \quad Y = XB$$

Solve for $B = \begin{bmatrix} a \\ b \end{bmatrix}$

$$B_{OLS} = (X^T X)^{-1} X^T Y = \begin{bmatrix} 2 \\ -\frac{2}{3} \end{bmatrix} \approx \begin{bmatrix} 2 \\ -0.67 \end{bmatrix}$$

- Yes, possibile to calculate in a least square sence.
- a and b as above, $a = 2$, $b = -\frac{2}{3}$
- The solution is the optimal ordinary least sqaures estimate, when $(X^T X)^{-1}$ exists!

e) Given linear equation

$$Y = \Theta Z$$

where Y and Z are known!

- Projection of Y onto Z is

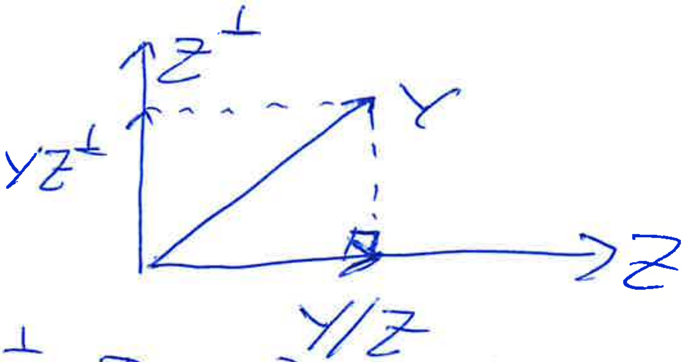
$$Y/Z = YZ^T(ZZ^T)^+ Z$$

Questions

- $$\Theta_{OLS} = YZ^T(ZZ^T)^{-1}$$

- Prediction of Y is

$$\bar{Y} = \Theta_{OLS} \cdot Z = YZ^T(ZZ^T)^{-1} Z = Y/Z$$

- 

Here $Z^\perp = I - Z^T(ZZ^T)^+ Z$

and $YZ^\perp = Y - YZ^T(ZZ^T)^+ Z$

Also $Y/Z + YZ^\perp = Y$

f) Define two integer parameters L and J
Define two matrix equations

$$Y_{J|L} = O_L X_J + H_L^d U_{J|L+g-1} + H_L^s E_{J|L} \quad (1)$$

$$Y_{J+1|L} = \tilde{A}_L Y_{J|L} + \tilde{B}_L U_{J|L+g} + \tilde{C}_L E_{J|L+1} \quad (2)$$

Define the instrument matrix W such that

$$W = \begin{bmatrix} U_{J|L+g} \\ U_{0|J} \\ Y_{0|J} \end{bmatrix}$$

We have that

$$E_{J|L} / W = 0$$

$$E_{J+1|L} / W = 0$$

when no feedback in data, i.e $E_{J|L+g}$ not correlated with $U_{J|L+g}$.

Remove noise

$$Y_{J|L}/W = O_L X_J/W + H_L^d U_{J|L+g-1} \quad (3)$$

$$Y_{J+1|L}/W = \tilde{A}_L Y_{J|L}/W + \tilde{B}_L U_{J|L+g} \quad (4)$$

Remove input term $U_{J|L+g-1}$ from (3)

$$(Y_{J|L}/W)\,U_{J|L+g}^{\perp} = O_L X_J^{a}$$

We have

$$Z_{J|L} = \left(Y_{J|L} \Big/ \begin{bmatrix} U_{J|L+g} \\ U_{0|J} \\ Y_{0|J} \end{bmatrix}\right) U_{J|L+g}^{\perp} \qquad (5)$$

where $U_{J|L+g}^{\perp} = I - U_{J|L+g}^{T}(U_{J|L+g}U_{J|L+g}^{T})^{+}U_{J|L+g}$

Take the SVD of (5)

$$Z_{J|L} = USV^{T} = [U_1\ U_2]\begin{bmatrix} S_1 & 0 \\ 0 & S_2 \end{bmatrix}[V_1\ V_2]^{T}$$
$$= U_1S_1V_1^{T} + U_2S_2V_2^{T}$$

Putting zero/smaller singular values in S_2 gives

n = the large singular values in S_1

$$Z_{J|L} = U_1 S_1 V_1^{T} = O_L X_J^{a}$$

Chose $U_1 = O_L$ and $X_J^{a} = S_1V_1^{T}$

$$\underline{O_L = U_1}$$