University of South-Eastern Norway

EXAMINATION INFORMATION PAGE
Written examination

Subject code: **IIA2217**	Subject name: System Identification and Optimal Estimation	
Examination date: June 6th 2019	Examination time from/to: 9-14	Total hours: 5
Responsible subject teacher: David Di Ruscio		
Campus: Porsgrunn	Faculty: Faculty of Technology	
No. of assignments: 4	No. of attachments: 0	No. of pages incl. front page and attachments: 6
Permitted aids: **Pen and paper**		
Information regarding attachments:		
Comments:		

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Task 1 (30%): Kalman Filter, Prediction Error Method

Given a linear system

$$\begin{aligned} x_{k+1} &= Ax_k + Bu_k + v_k, \quad (1) \\ y_k &= Dx_k + w_k, \quad (2) \end{aligned}$$

where v_k and w_k are white process and measurements noise, respectively.

a) What is a Kalman filter ? (give a short description)

b) Formulate a Kalman filter on *apriori-aposteriori* form for the linear system in Eqs. (1)-(2), i.e. involving variables $\bar{x}_k$ and $\hat{x}_k$ and $\bar{y}_k$.

c) Formulate the *innovations formulation* of the Kalman filter.

d) Commenting upon the relationship between the two formulations in 2) and 3) above.

e) Given a non-linear system

$$\begin{aligned} x_{k+1} &= f(x_k, u_k) + v_k \quad (3) \\ y_k &= g(x_k) + w_k \quad (4) \end{aligned}$$

where v_k and w_k are discrete white process noise and discrete white measurements noise, respectively.

Formulate the Kalman filter on apriori-aposteriori form for the non-linear system model in Eqs. (3) and (4).

f) The Kalman-filter on prediction form is often used in connection with Prediction error Methods (PEM) for system identification.

- Write down the Kalman filter on prediction form for the system in Eqs. (1) and (2) ?
- Define the Prediction Error (PE) ε_k ?
- Assume that we want to use PEM to find the parameters in a single input single output system with $n = 2$ states. Write down the structure of a 2nd order model on canonical form and define the parameter vector θ ?

Task 2 (30%): Linear Regression and realization theory

a) Given a 1st order system (ie. a system with one state)

$$x_{k+1} = \theta_1 x_k + \theta_2 u_k + \theta_1 e_k, \quad (5)$$

$$y_k = x_k + e_k, \quad (6)$$

where θ_1 and θ_2 are unknown parameters and e_k is white noise.

Find a linear regression model of the form

$$y_k = \varphi_k^T \theta_0 + e_k. \quad (7)$$

Define the parameter vector θ_0 and the vector φ_k of regressors.

b) Consider the linear regression model Eq. (7) and known data variables y_k and φ_k for known discrete time instants $k = 1, \ldots, N$.

Consider the following prediction error criterion

$$V_N(\theta) = \frac{1}{N} \sum_{k=1}^{N} \varepsilon_k^T \Lambda \varepsilon_k \quad (8)$$

where Λ is a specified and symmetric weighting matrix.

- Define the Prediction Error ε_k ?
- Find the Ordinary Least Squares (OLS) estimate, $\hat{\theta}_N$, of the true parameter vector θ_0.

c) Orthogonal projections. Consider the linear equation

$$Y = OX + E, \quad (9)$$

where the matrices $Y \in \mathbb{R}^{m \times N}$ and $X \in \mathbb{R}^{n \times N}$ are known matrices. The matrix $E \in \mathbb{R}^{m \times N}$ is a matrix of noise and uncorrelated with X. We assume $N > n$.

- Define the mathematical definitions for the orthogonal projections Y/X and $YX^{\perp}$?
- Write down a two dimensional figure to illustrate the projections?
- Is $Y = Y/X + YX^{\perp}$ correct or not ?

d) Assume known impulse responses

$$H_k = DA^{k-1}B \ \forall \ k = 1, \cdots, 5. \quad (10)$$

Answer the following:

- Write down the Hankel matrices $\mathbf{H}_{1|L}$ and $\mathbf{H}_{2|L}$ where you should use $L = 2$.
- How are $\mathbf{H}_{1|L}$ and $\mathbf{H}_{2|L}$ related to O_L, C_J and A ? (Here O_L and C_J are the extended observability and controllability matrices, respectively.)
- Show how you can find the system order, n, the extended observability matrix O_L and the extended controllability matrix C_J from a Singular Value decomposition (SVD).
- Find a formula for calculating the system matrix A ?

Task 3 (20%): Autonomous systems

Assume given measured outputs

$$y_k \; \forall \; k = 0, \ldots, N-1 \tag{11}$$

from an autonomous system described by the model

$$x_{k+1} = Ax_k, \; (x_0 \neq 0 \text{ The initial state}) \tag{12}$$

$$y_k = Dx_k, \tag{13}$$

where $x_k \in \mathbb{R}^n$ is the state vector and x_0 is the initial state vector at the initial time instant $k = 0$.
The **problem** in this task is to identify the system order n, the model matrices A and D as well as the initial state vector x_0 !

a) Matrix equations:

- Write down one matrix equation involving the Hankel matrix $Y_{0|L}$, the extended Observability matrix O_L and a matrix X_0 with states.
- Write down one matrix equation involving the Hankel matrix $Y_{1|L}$, the extended Observability matrix O_L, a matrix X_0 with states and the system matrix A.
- Define the structure of the matrices $Y_{0|L}$, $Y_{1|L}$ and X_0.
- How may we analyze whether the system in eq. (12) is stable or not?

b) Describe how n, O_L and X_0 may be estimated from a Singular Value Decomposition (SVD) of one of the Hankel matrices $Y_{0|L}$ or $Y_{1|L}$?

c) Find a formula for estimating/calculating the system matrix A ?

d) How can we find estimates of the output matrix D and the initial state vector x_0 ?

Task 4 (20%): Subspace System Identification: Deterministic and combined systems with feedback in data

Consider the discrete time deterministic model, ie.

$$
\begin{aligned}
x_{k+1} &= Ax_k + Bu_k, && (14)\\
y_k &= Dx_k + Eu_k, && (15)
\end{aligned}
$$

where the following output and input data matrices are known

$$
Y = \begin{bmatrix} y_0^T \\ y_1^T \\ \vdots \\ y_{N-1}^T \end{bmatrix} \in \mathbb{R}^{N\times m}, \quad U = \begin{bmatrix} u_0^T \\ u_1^T \\ \vdots \\ u_{N-1}^T \end{bmatrix} \in \mathbb{R}^{N\times r}. \qquad (16)
$$

a) Based on the model in Equations (14) and (15) and with known data as given in (16) we can develop the following matrix equations

$$
\begin{aligned}
Y_{0|L} &= O_L X_0 + H_L^d U_{0|L+g-1}, && (17)\\
Y_{1|L} &= \tilde{A}_L Y_{0|L} + \tilde{B}_L U_{0|L+g}, && (18)
\end{aligned}
$$

where $L \geq 1$ is a user specified positive integer.

- Write down the structure of the matrices in the matrix equations, (17) and (18), with parameters $N = 10$, $L = 2$, $J = 2$ and $g = 1$.
- Write down the expressions for the matrices $\tilde{A}_L$ and $\tilde{B}_L$!

b) By using (16) and Equations (17) and (18) we may formulate the equations

$$
Z_{0|L} = O_L X_0^a \qquad (19)
$$

and

$$
Z_{1|L} = \tilde{A}_L Z_{0|L} \qquad (20)
$$

Find expressions for the data matrices $Z_{0|L}$ and $Z_{1|L}$.

Remark: define the projections which is involved in the expressions for $Z_{1|L}$ and $Z_{0|L}$.

c)

- Show how the system matrix A can be estimated ?
- Describe shortly how the matrices B and E may be estimated ?

d) Consider a combined deterministic and stochastic system described by the following model on innovations form, ie.

$$x_{k+1} = Ax_k + Bu_k + Ce_k, \quad (21)$$
$$y_k = Dx_k + Fe_k \quad (22)$$

where e_k is white noise with unit covariance matrix, i.e., $\mathrm{E}(e_k e_k^T) = I$ and where the output and input data matrices as in Eq. (16) are known

Consider that the known input and output data as given in (16) are collected in closed loop, i.e., we assume that there is feedback in the known data.

From Eq. (22) we may define the matrix Equation

$$Y_{J|1} = DX_{J|1} + FE_{J|1} \quad (23)$$

- Define the Hankel matrices $Y_{J|1}$, $X_{J|1}$ and $E_{J|1}$.
  When $J \to \infty$ we can prove that the following identities holds

$$X_{J|1} = X_{J|1} / \begin{bmatrix} U_{0|J} \\ Y_{0|J} \end{bmatrix} \quad (24)$$

and

$$E_{J|1} / \begin{bmatrix} U_{0|J} \\ Y_{0|J} \end{bmatrix} = 0. \quad (25)$$

Use (24), (25) and (23) to find a projection

$$Z^s_{J|1} = FE_{J|1} \quad (26)$$

such that the innovations sequence

$$\begin{aligned} Z^s_{J|1} &= \begin{bmatrix} Fe_J & Fe_{J+1} & \dots & Fe_{N-1} \end{bmatrix} \\ &= \begin{bmatrix} \varepsilon_J & \varepsilon_{J+1} & \dots & \varepsilon_{N-1} \end{bmatrix} \end{aligned} \quad (27)$$

is estimated and hence could be considered as known.

- Explain how this projection can be used in order to develop a subspace identification algorithm for closed loop systems?