University of South-Eastern Norway

EXAMINATION INFORMATION PAGE
Written examination

Subject code: **IIA2217**	Subject name: System Identification and Optimal Estimation	
Examination date: June 7, 2018	Examination time from 9-14	Total hours: 5
Responsible subject teacher: David Di Ruscio. Phone 40751996		
Campus: Porsgrunn	Faculty: Faculty of Technology, Natural Sciences and Maritime Sciences	
No. of assignments: 5	No. of attachments: 0	No. of pages incl. front page and attachments: 6
Permitted aids: Pen and paper		
Information regarding attachments:		
Comments:		

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Task 1 (20%): Autonomous systems

Assume given measured outputs

$$y_k \ \forall \ k = 0, \dots, N-1 \tag{1}$$

from an autonomous system described by the model

$$x_{k+1} = Ax_k, \ (x_0 \neq 0 \text{ The initial state}) \tag{2}$$
$$y_k = Dx_k, \tag{3}$$

where $x_k \in \mathbb{R}^n$ is the state vector and x_0 is the initial state vector at the initial time instant $k = 0$.
The **problem** in this task is to identify the system order n, the model matrices A and D as well as the initial state vector x_0 !

a) Matrix equations:

- Write down one matrix equation involving the Hankel matrix $Y_{0|L}$, the extended Observability matrix O_L and a matrix X_0 with states.
- Write down one matrix equation involving the Hankel matrix $Y_{1|L}$, the extended Observability matrix O_L, a matrix X_0 with states and the system matrix A.
- Define the structure of the matrices $Y_{0|L}$, $Y_{1|L}$ and X_0.

b) Describe how n, O_L and X_0 may be estimated from a Singular Value Decomposition (SVD) of the Hankel matrix $Y_{0|L}$.

c) Find a formula for estimating/calculating the system matrix A ?

d) How can we find estimates of the output matrix D and the initial state vector x_0 ?

Task 2 (20%): Deterministic Subspace System Identification

Consider the discrete time deterministic model, ie.

$$x_{k+1} = Ax_k + Bu_k, \tag{4}$$
$$y_k = Dx_k + Eu_k, \tag{5}$$

where the following output and input data matrices are known

$$Y = \begin{bmatrix} y_0^T \\ y_1^T \\ \vdots \\ y_{N-1}^T \end{bmatrix} \in \mathbb{R}^{N\times m}, \ U = \begin{bmatrix} u_0^T \\ u_1^T \\ \vdots \\ u_{N-1}^T \end{bmatrix} \in \mathbb{R}^{N\times r}. \tag{6}$$

a) Based on the model in Equations (4) and (5) and with known data as given in (6) we can develop the following matrix equations

$$Y_{0|L} = O_L X_0 + H_L^d U_{0|L+g-1}, \tag{7}$$

$$Y_{1|L} = \tilde{A}_L Y_{0|L} + \tilde{B}_L U_{0|L+g}, \tag{8}$$

where $L \geq 1$ is a user specified positive integer.

- Write down the structure of the matrices in the matrix equations, (7) and (8), with parameters $N = 10$, $L = 2$, $J = 2$ and $g = 0$.
- Write down the expressions for the matrices $\tilde{A}_L$ and $\tilde{B}_L$!

b) By using (6) and Equations (7) and (8) we may formulate the equations

$$Z_{0|L} = O_L X_0^a \tag{9}$$

and

$$Z_{1|L} = \tilde{A}_L Z_{0|L} \tag{10}$$

Find expressions for the data matrices $Z_{0|L}$ and $Z_{1|L}$.

Remark: define the projections which is involved in the expressions for $Z_{1|L}$ and $Z_{0|L}$.

c) Show how

- the system matrix A

can be estimated.

d) Assume that the system is single output. Is it then possible to write the deterministic system as a linear regression model? , i.e., as a model

$$y_k = \varphi_k^T \theta \tag{11}$$

where φ_k contains the regressors and θ is a vector of parameters. The answer is YES? or NO?

Task 3 (20%): Prediction error methods

A Kalman filter on innovations form for a linear discrete time system is given by

$$\bar{x}_{k+1} = A\bar{x}_k + Bu_k + Ke_k, \tag{12}$$
$$y_k = D\bar{x}_k + e_k \tag{13}$$

where $\bar{x}_k$ is the predicted state, $\bar{x}_1$ is the initial state, $y_k \in \mathbb{R}^m$ is the measurement vector and e_k is the innovations process.
We will assume that the following input and output data are known

$$\left.\begin{matrix} u_k \\ y_k \end{matrix}\right\} \;\forall\; k = 1, \ldots, N \tag{14}$$

a) Write down a Kalman filter on prediction form, i.e. the filter used to compute the predicted measurement, $\bar{y}_k$, of the measurement y_k.

b)

- What is the meaning of a parameter vector, θ?
- Give an example of the relationship between the parameter vector θ and the prediction formulation of a Kalman filter for a single output dynamic system in state space as in (12) and (13) with $n = 3$ states.
- How many parameters, p, is it in this parameter vector, $\theta \in \mathbb{R}^p$?

c) Define the prediction error, ε_k, as a function of y_k and $\bar{y}_k$.

d) Define and answer the following questions:

- What is a prediction error criterion, $V_N(\theta)$.
- Give an example of a prediction error criterion for both a single output system ($m = 1$) and a multiple output system ($m > 1$.
- Describe how the optimal parameter estimate, $\hat{\theta}_N$, can be computed.

Task 4 (20%): Ordinary Least Squares method and recursive system identification

a) Given a system

$$\begin{aligned} x_{k+1} &= ax_k + bu_k + ke_k, && (15) \\ y_k &= x_k + e_k. && (16) \end{aligned}$$

Write the model as an ARX model and hence as a linear regression model of the form

$$y_k = \varphi_k^T \theta_0 + e_k \tag{17}$$

- In particular, define the regression vector, φ_k, and the parameter vector, θ_0.
- Also define the parameter k for this to be true ?.

b)

- Based on the regression model in step a) above, find a predictor, $\bar{y}_k(\theta)$, for the measurement y_k.
- Define the prediction error, ε_k.

c) Consider the following prediction error criterion

$$V_N(\theta) = \frac{1}{N} \sum_{k=1}^{N} \varepsilon_k^T \Lambda \varepsilon_k \tag{18}$$

where Λ is a specified and symmetric weighting matrix.

Find the Ordinary Least Squares (OLS) estimate, $\hat{\theta}_N$, of the true parameter vector θ_0.

d) Based on the OLS solution in step c) above, show how we can develop a recursive Ordinary Least Squares (ROLS) method of the following form

$$\hat{\theta}_t = \hat{\theta}_{t-1} + K_t(y_t - \varphi_t^T \hat{\theta}_{t-1}) \tag{19}$$

You shall in particular find equations for computing the gain, K_t, in Equation (19).

Task 5 (20%): Various questions

a) Given a system modelled by the linear model

$$Y = XB + E \tag{20}$$

where $Y \in \mathbb{R}^{N\times m}$ and $X \in \mathbb{R}^{N\times r}$ are known data matrices $E \in \mathbb{R}^{N\times m}$ is a matrix of normally distributed white noise. We are assuming a large number of observations, N, such that $N > r$.

Find an Ordinary Least Squares (OLS) estimate, B_{OLS}, of the unknown matrix of regression coefficients, B.

b) Assume that the data matrix X is not of full column rank.

- Show how you can perform a principal Component Analysis (PCA) of the X matrix by using a Singular Value Decomposition (SVD). What is the number of Principal Components, a.
- Find a Principal Component Regression (PCR) estimate, B_{PCR}, of the unknown matrix of regression coefficients, B.

c) You should in this sub task find a state space model realization, i.e., obtain the state space model matrices (A, B, D), from known impulse responses matrices, by using Hankel matrix realization theory. I.e. answer the following:

- Write down the Hankel matrices $\mathbf{H}_{1|L}$ and $\mathbf{H}_{2|L}$ where you should use $L = 2$.
- Show how you can find the system order, n, the extended observability matrix O_L and the extended controllability matrix C_L from a Singular Value decomposition (SVD) of $\mathbf{H}_{1|L}$.
- Show how the D and B matrices can be found.
- Show how the A matrix can be found.

d) Given a system modelled by a discrete time, state space model as follows

$$x_{k+1} = Ax_k + Bu_k + v_k, \tag{21}$$

$$y_k = Dx_k + Eu_k + w_k, \tag{22}$$

where v_k is white process noise and w_k is white measurements noise.

- Write down a Kalman filter on apriori-aposteriori form for optimal estimation of the state vector x_k.
- Show how the apriori-aposteriori formulation of the Kalman filter kan be written as a Kalman filter on innovations form.