

Final Exam SCE2206
**System identification and optimal
estimation**

Thursday June 8, 2017

Time: kl. 9.00 - 14.00

The final exam consists of: 5 tasks.
The exam counts 100% of the final grade.

Available aids: pen and paper

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Teacher: PhD David Di Ruscio
Systems and Control Engineering
Faculty of Technology
Telemark University College
N-3914 Porsgrunn

Task 1 (40%): Model structures and ordinary least squares regression

For simplicity in the following subtask we assume SISO systems.

- What is an ARMAX model ?
- What is an ARX model ?
- Given a deterministic linear state space model

$$x_{k+1} = Ax_k + Bu_k, \quad (1)$$

$$y_k = Dx_k. \quad (2)$$

Can we write this model as an ARX model or an ARMAX model ?

- Given a Kalman filter on innovations form

$$x_{k+1} = Ax_k + Bu_k + Ke_k, \quad (3)$$

$$y_k = Dx_k + e_k. \quad (4)$$

Can we write this model as an ARX model or an ARMAX model ?

- Given a 1st order ($n = 1$) state model

$$x_{k+1} = ax_k + bu_k + ke_k, \quad (5)$$

$$y_k = x_k + e_k. \quad (6)$$

where a , b and $k = a$ are model parameters. Introduce the backward shift operator q^{-1} such that $q^{-1}y_k = y_{k-1}$

- Write this model on input and output polynomial form and decide if it is an ARX or ARMAX model ?

- Given a 1st order system (ie. a system with one state)

$$x_{k+1} = \theta_1 x_k + \theta_2 u_k + \theta_1 e_k, \quad (7)$$

$$y_k = x_k + e_k, \quad (8)$$

where θ_1 and θ_2 are unknown parameters and e_k is white noise.

Find a linear regression model of the form

$$y_k = \varphi_k^T \theta + e_k. \quad (9)$$

Define the parameter vector θ and the vector φ_k of regressors.

g) Consider the following prediction error criterion

$$V_N(\theta) = \frac{1}{N} \sum_{k=1}^N \varepsilon_k^T \Lambda \varepsilon_k \quad (10)$$

where $\Lambda > 0$ is a specified and symmetric weighting matrix.

- Find the Ordinary Least Squares (OLS) estimate, $\hat{\theta}_N$, of the true parameter vector θ_0 .
- Does there exist an optimal weighting matrix Λ ? If so, what is the name of the corresponding estimate and the optimal Λ ?

Task 2 (20%): Realization theory, Kalman filter, Prediction error Method

a) Assume known impulse responses

$$H_k = DA^{k-1}B \quad \forall k = 1, \dots, L + J. \quad (11)$$

Answer the following:

- Write down the Hankel matrices $\mathbf{H}_{1|L}$ and $\mathbf{H}_{2|L}$ where you should use $L = 3$ and $J = 4$.
- What is the interpretation of the parameters L and J ?
- How are $\mathbf{H}_{1|L}$ and $\mathbf{H}_{2|L}$ related to O_L , C_J and A ? Define O_L and C_J , i.e. the extended observability and controllability matrices, respectively.
- Show how you can find the system order, n , the extended observability matrix O_L and the extended controllability matrix C_J from a Singular Value decomposition (SVD).
- Find a formula for calculating the system matrix A ?

b) Given a system modelled by a discrete time, state space model as follows

$$x_{k+1} = Ax_k + Bu_k + v_k, \quad (12)$$

$$y_k = Dx_k + Eu_k + w_k, \quad (13)$$

where v_k is white process noise and w_k is white measurements noise. Assume that the noise are uncorrelated, i.e. $E(v_k w_k^T) = 0$.

- Write down a Kalman filter on a priori-posteriori form for optimal estimation of the state vector x_k .

- Find a formula for the Kalman filter gain matrix, K , in the Kalman filter on apriori-aposteriori form.
- Show how the apriori-aposteriori formulation of the Kalman filter can be written as a Kalman filter on innovations form.

c) Given a non-linear system

$$x_{k+1} = f(x_k, u_k) + v_k \quad (14)$$

$$y_k = g(x_k) + w_k \quad (15)$$

where v_k and w_k are discrete white process noise and discrete white measurements noise, respectively.

Formulate the Kalman filter on apriori-aposteriori form for the non-linear system model in Eqs. (14) and (15).

d) The Kalman-filter on prediction form is often used in connection with Prediction error Methods (PEM) for system identification.

- Write down the Kalman filter on prediction form for the system in Eqs. (12) and (13) ?
- Define the Prediction Error (PE) ε_k ?
- Assume that we want to use PEM to find the parameters in a single input single output system with $n = 2$ states. Write down the structure of a 2nd order model on canonical form and define the parameter vector θ ?

Task 3 (20%): Autonomous systems

Assume given measured outputs

$$y_k \quad \forall k = 0, \dots, N - 1 \quad (16)$$

from an autonomous system described by the model

$$x_{k+1} = Ax_k, \quad (x_0 \neq 0 \text{ The initial state}) \quad (17)$$

$$y_k = Dx_k, \quad (18)$$

where $x_k \in \mathbb{R}^n$ is the state vector and x_0 is the initial state vector at the initial time instant $k = 0$.

The **problem** in this task is to identify the system order n , the model matrices A and D as well as the initial state vector x_0 !

a) Matrix equations:

- Write down one matrix equation involving the Hankel matrix $Y_{0|L}$, the extended Observability matrix O_L and a matrix X_0 with states.
- Write down one matrix equation involving the Hankel matrix $Y_{1|L}$, the extended Observability matrix O_L , a matrix X_0 with states and the system matrix A .
- Define the structure of the matrices $Y_{0|L}$, $Y_{1|L}$ and X_0 .
- What is the importance of the matrix A in eq. (17) ? Explain some properties !

b) Describe how n , O_L and X_0 may be estimated from a Singular Value Decomposition (SVD) of one of the Hankel matrices $Y_{0|L}$ or $Y_{1|L}$?

c) Find a formula for estimating/calculating the system matrix A ?

d) How can we find estimates of the output matrix D and the initial state vector x_0 ?

Task 4 (20%): Subspace System Identification: Deterministic and combined systems with feedback in data

Consider the discrete time deterministic model, ie.

$$x_{k+1} = Ax_k + Bu_k, \quad (19)$$

$$y_k = Dx_k + Eu_k, \quad (20)$$

where the following output and input data matrices are known

$$Y = \begin{bmatrix} y_0^T \\ y_1^T \\ \vdots \\ y_{N-1}^T \end{bmatrix} \in \mathbb{R}^{N \times m}, \quad U = \begin{bmatrix} u_0^T \\ u_1^T \\ \vdots \\ u_{N-1}^T \end{bmatrix} \in \mathbb{R}^{N \times r}. \quad (21)$$

a) Based on the model in Equations (19) and (20) and with known data as given in (21) we can develop the following matrix equations

$$Y_{0|L} = O_L X_0 + H_L^d U_{0|L+g-1}, \quad (22)$$

$$Y_{1|L} = \tilde{A}_L Y_{0|L} + \tilde{B}_L U_{0|L+g}, \quad (23)$$

where $L \geq 1$ is a user specified positive integer.

- Write down the structure of the matrices in the matrix equations, (22) and (23), with parameters $N = 10$, $L = 2$, $J = 2$ and $g = 1$.
 - Write down the expressions for the matrices \tilde{A}_L and \tilde{B}_L !
- b) By using (21) and Equations (22) and (23) we may formulate the equations

$$Z_{0|L} = O_L X_0^a \quad (24)$$

and

$$Z_{1|L} = \tilde{A}_L Z_{0|L} \quad (25)$$

Find expressions for the data matrices $Z_{0|L}$ and $Z_{1|L}$.

Remark: define the projections which is involved in the expressions for $Z_{1|L}$ and $Z_{0|L}$.

- c)
- Show how the system matrix A can be estimated ?
 - Describe shortly how the matrices B and E may be estimated ?
- d) Consider a combined deterministic and stochastic system described by the following model on innovations form, ie.

$$x_{k+1} = Ax_k + Bu_k + Ce_k, \quad (26)$$

$$y_k = Dx_k + Fe_k \quad (27)$$

where e_k is white noise with unit covariance matrix, i.e., $E(e_k e_k^T) = I$ and where the output and input data matrices as in Eq. (21) are known

Consider that the known input and output data as given in (21) are collected in closed loop, i.e., we assume that there is feedback in the known data.

From Eq. (27) we may define the matrix Equation

$$Y_{J|1} = DX_{J|1} + FE_{J|1} \quad (28)$$

- Define the Hankel matrices $Y_{J|1}$, $X_{J|1}$ and $E_{J|1}$.

When $J \rightarrow \infty$ we can prove that the following identities holds

$$X_{J|1} = X_{J|1} / \begin{bmatrix} U_{0|J} \\ Y_{0|J} \end{bmatrix} \quad (29)$$

and

$$E_{J|1} / \begin{bmatrix} U_{0|J} \\ Y_{0|J} \end{bmatrix} = 0. \quad (30)$$

Use (29), (30) and (28) to find a projection

$$Z_{J|1}^s = FE_{J|1} \quad (31)$$

such that the innovations sequence

$$\begin{aligned} Z_{J|1}^s &= [Fe_J \quad Fe_{J+1} \quad \dots \quad Fe_{N-1}] \\ &= [\varepsilon_J \quad \varepsilon_{J+1} \quad \dots \quad \varepsilon_{N-1}] \end{aligned} \quad (32)$$

is estimated and hence could be considered as known.

- Explain how this projection can be used in order to develop a subspace identification algorithm for closed loop systems?