Task 1 (SISO systems assumed)

a) An input-output polynomial model

$$A(q)\,y_k = B(q)\,u_k + C(q)\,e_k$$

where $A(q)$, $B(q)$ and $C(q)$ are polynomials in the shift operator q^{-1}. (ARMAX)

Auto Regressive Moving Average with eXtra (eXogenous) variables

b) An input-output polynomial model

$$A(q)\,y_k = B(q)\,u_k + e_k$$

Auto Regressive with eXtra variables - ARX

c) A deterministic model $\begin{aligned} x_{k+1} &= Ax_k + Bu_k \\ y_k &= Dx_k \end{aligned}$

may be written as an ARX model

d) A more general state space model $\begin{cases} x_{k+1} = Ax_k + Bu_k + Ke_k \\ y_k = Dx_k + e_k \end{cases}$ may be written as an ARMAX model.

e)

e)

$$x_k = a x_{k-1} + b u_{k-1} + k e_{k-1}$$

and $x_k = y_k - e_k$ gives

$$y_k - e_k = a(y_{k-1} - e_{k-1}) + b u_{k-1} + k e_{k-1}$$

$$y_k - a y_{k-1} = b u_{k-1} + e_k + (k-a) e_{k-1}$$

gives

$$\overbrace{(1 - a q^{-1})}^{A(q)} y_k = \overbrace{b q^{-1}}^{B(q)} u_k + \overbrace{(1 - (k-a) q^{-1})}^{C(q)} e_k$$

where $q^{-1} y_k = y_{k-1}$, $q^{-1} u_k = u_{k-1}$, and $q^{-1} e_k = e_{k-1}$.

With $k = a$ we have $C(q) = 1$ and an ARX model

f)

$\Rightarrow$

4f)

$$x_k = \theta_1 x_{k-1} + \theta_2 u_{k-1} + \theta_1 e_{k-1}$$

and $x_k = y_k - e_k$ gives

$$\underline{y_k} - e_k = \theta_1 (\underline{y_{k-1}} - \cancel{e_{k-1}}) + \underline{\theta_2 u_{k-1}} \cancel{+ \theta_1 e_{k-1}}$$

$$\Downarrow$$

$$y_k = \theta_1 y_{k-1} + \theta_2 u_{k-1} + e_k = [y_{k-1}\ u_{k-1}] \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} + e_k$$

We write as linear regression model

$$y_k = \varphi_k^T \theta + e_k$$

where $\underline{\underline{\varphi_k^T = [y_{k-1}\ u_{k-1}], \theta = \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix}}}$

4g)

$$\hat{\theta}_N = \left(\sum_{k=1}^{N} \varphi_k \Lambda \varphi_k^T \right)^{-1} \sum_{k=1}^{N} \varphi_k \Lambda y_k$$

assuming $\varepsilon_k = y_k - \hat{y}_k(\theta) = y_k - \varphi_k^T \theta$

- When $y_k = \varphi_k^T \theta + e_k$ and $E(e_k e_k^T) = \Delta$.

Optimal weighting matrix

$$\Lambda = \Delta^{-1} = (E(e_k e_k^T))^{-1}$$

i.e. The BLUE, Best Linear Unbiased Estimate

Task 2

a) Given $L=3$, $J=4$ and $H_k = DA^{k-1}B$ $\forall k=1,\dots,7$. $N=L+J=7$

- $H_{2|4} = \begin{bmatrix} H_2 & H_3 & H_4 & H_5 & H_6 \\ H_3 & H_4 & H_5 & H_6 & H_7 \end{bmatrix}$, $H_{1|4} = \begin{bmatrix} H_1 & H_2 & H_3 & H_4 & H_5 \\ H_2 & H_3 & H_4 & H_5 & H_6 \end{bmatrix}$
- $L=2$ number of ~~columns~~ rows in $H_{1|2}$ and $H_{2|2}$
  $J=4$ —"— of columns in —"—
- $H_{1|L} = O_L C_J$ and $H_{2|L} = O_L A C_J$

where $O_L = \begin{bmatrix} D \\ DA \\ \vdots \\ DA^{L-1} \end{bmatrix}$ – the extended observability matrix.

$C_J = [B \;\; AB \;\cdots\; A^{J-1}B]$ – $\left\{\begin{array}{l}\text{The extended} \\ \text{controllability} \\ \text{matrix}\end{array}\right.$

– Related to def. of L
System order bounded by

$$1 \leq n \leq L \cdot m$$

where $\dim(y_k) = m$, $D \in \mathbb{R}^{m \times n}$

2 b) Initial value for $\bar{x}_{k=0}$ at startup. Constant K.

- $\bar{y}_k = D\bar{x}_k + Eu_k$ – Predicted output

  $\varepsilon_k = y_k - \bar{y}_k$ – Innovations

  $\hat{x}_k = \bar{x}_k + K\varepsilon_k$ – $\begin{cases} \text{Aposteriori estimate} \\ K\text{-Kalman gain} \end{cases}$

  $\bar{x}_{k+1} = A\hat{x}_k + Bu_k$ – update apriori state estimate

- Varying Kalman gain matrix or constant

$$K = \bar{X}D^T(D\bar{X}D^T + W)^{-1}$$

where $\bar{X} = E((x-\bar{x})(x-\bar{x})^T)$

- May eliminat $\hat{x}_i$ and we obtain

$$\left.\begin{aligned} \bar{x}_{k+1} &= A\bar{x}_k + Bu_k + \tilde{K}\varepsilon_k \\ y_k &= D\bar{x}_k + Eu_k + \varepsilon_k \end{aligned}\right\} (1)$$

whee $\tilde{K} = AK$

the Kalman gain matrix in inovations form (1).

- SVD of $H_{1|L} = O_L C_J$

$$H_{1|L} = U S V^T = [U_1 \; U_2] \begin{bmatrix} S_1 & 0 \\ 0 & S_2 \end{bmatrix} [V_1 \; V_2]^T$$

$S_2 \approx 0$

$$= U_1 S_1 V_1^T$$

- system order n = to number of non-zero singular values

$$S_1 = \begin{bmatrix} s_1 & & 0 \\ & \ddots & \\ 0 & & s_n \end{bmatrix}, \quad S_2 \approx 0$$

- $O_L = U_1 S_1$ and $C_J = V_1^T$ gives output normal realization

- Solve A from $H_{2|L} = O_L A C_J$ gives

$$A = (O_L^T O_L)^{-1} O_L^T H_{2|L} C_J^T (C_J C_J^T)^{-1}$$

or using SVD, Solve $H_{2|L} = U_1 S_1 A V_1^T$

$U_1^T H_{2|L} V_1 = S_1 A \Rightarrow A = S_1^{-1} U_1^T H_{2|L} V_1$

2c)

- For simplicity, constant K

Initial predicted state $\bar{x}_{k=0}$

$$(1)\begin{cases} \bar{y}_k = g(\bar{x}_k) \\ \hat{x}_k = \bar{x}_k + K\varepsilon_k, \quad \varepsilon_k = y_k - \bar{y}_k \\ \bar{x}_{k+1} = f(\hat{x}_k, u_k) \end{cases}$$

- with varying K, change (1) above

– Initial values for $\bar{x}_{k=0}$ and $\bar{X}_{k=0}$

$$\bar{y}_k = g(\bar{x}_k)$$

$$D_k = \left.\frac{d\, g(x_k, u_k)}{d x_k^T}\right|_{\bar{x}_k, u_k}$$

$$K_k = \bar{X}_k D_k^T (D_k \bar{X}_k D_k^T + W)^{-1}$$

$$\hat{x}_k = \bar{x}_k + K\varepsilon_k, \quad \varepsilon_k = y_k - \bar{y}_k$$

$$\bar{x}_{k+1} = f(\hat{x}_k, u_k)$$

$$\hat{X}_k = (I - K_k D_k)\bar{X}_k (I - K_k D_k)^T + K_k W K_k^T$$

$$\bar{X}_{k+1} = A\hat{X}_k A^T + V$$

Here $V = E(v_k v_k^T)$ and $W = E(w_k w_k^T)$ the covariance matrices of v_k and w_k, respectively

2a) Kalman filter on prediction form

- $$\bar{y}_k = D\bar{x}_k + Eu_k$$

$$\varepsilon_k = y_k - \bar{y}_k$$

$$\bar{x}_{k+1} = A x_k + B u_k + \bar{K}\overbrace{(y_k - \bar{y}_k)}^{\varepsilon_k}$$

or

$$\bar{x}_{k+1} = A\bar{x}_k + Bu_k + K(y_k - \bar{y}_k)$$
$$\bar{y}_k = D\bar{x}_k + Eu_k$$

- Prediction Error (PE)

$$\varepsilon_k = y_k - \bar{y}_k = y_k - D\bar{x}_k - Eu_k$$

- $n = 2$

$$A = \begin{bmatrix} 0 & 1 \\ \theta_1 & \theta_2 \end{bmatrix}, \quad B = \begin{bmatrix} \theta_3 \\ \theta_4 \end{bmatrix}, \quad K = \begin{bmatrix} \theta_5 \\ \theta_6 \end{bmatrix}$$

$$D = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad E = \theta_7, \quad \bar{x}_{k=0} = \begin{bmatrix} \theta_8 \\ \theta_9 \end{bmatrix}$$

$$\theta = [\theta_1, \theta_2 \cdots \theta_9]^T$$

Assuming SISO system!

Task 3

a) • $Y_{0|L} = O_L X_0$

whee

$$Y_{0|L} = \begin{bmatrix} y_0 & \cdots & & x \\ y_1 & \cdots & & x \\ \vdots & & & \\ y_{L-1} & y_L & \cdots & y_{N-2} \end{bmatrix}$$

$$X_0 = [x_0 \; x_1 \; \cdots \; x_{N-2}]$$

• $Y_{1|L} = O_L A X_0$

when

$$Y_{1|L} = \begin{bmatrix} y_1 & \cdots & & x \\ y_2 & \cdots & & x \\ \vdots & & & \\ y_L & y_{L+1} & \cdots & y_{N-1} \end{bmatrix}$$

• - A - the transition (system) matrix.
- eigenvalues of A defines stability, time constants.

b) O_L and X_0 estimated from SVD of $Y_{0|L} = U_1 S_1 V_1^T$ and $O_L = U_1 S_1$ and $X_0 = V_1^T$. System order n = # of non zero singular values of $Y_{0|L}$.

c) Solve $Y_{1|L} = U_1 S_1 A V_1^T \Rightarrow A = S_1^{-1} U_1^T Y_{1|L} V_1$

c) or solve

$$Y_{1|L} = O_L A X_0$$

$$\Downarrow$$

$$A = O_L^+ Y_{1|L} X_0^+$$

where $O_L^+ = (O_L^T O_L)^{-1} O_L^T$ – left pseudo inverse

$X_0^+ = X_0^T (X_0 X_0^T)^{-1}$ – right pseudo inverse

d) • Initial state vector x_0, 1st column in $X_0 = V_1^T = [x_0 \; x_1 \; \ldots]$

• D first block row in $O_L = U_1 S_1$

Task 4

a) $N=10$, $L=2$, $J=2$, $g=1$, $\left.\begin{matrix} u_k \\ y_k \end{matrix}\right\} \forall k=0,1,\dots,9$

Remark, $J=2$ not used in Eq. (22)-(23) and not used information

$$Y_{1|L} = Y_{1|2} = \begin{bmatrix} y_1 & y_2 & \cdots & y_8 \\ y_2 & y_3 & \cdots & y_9 \end{bmatrix}, \quad Y_{0|2} = \begin{bmatrix} y_0 & y_1 & \cdots & y_7 \\ y_1 & y_2 & \cdots & y_8 \end{bmatrix}$$

$$U_{0|L+g} = U_{0|3} = \begin{bmatrix} u_0 & \cdots & \cdots & u_7 \\ u_1 & \cdots & \cdots & u_8 \\ u_2 & u_3 & \cdots & u_9 \end{bmatrix}$$

Remark

$K = 8$ columns!

$$O_L = O_2 = \begin{bmatrix} D \\ DA \end{bmatrix} \quad H_2^d = \begin{bmatrix} E & 0 \\ DB & E \end{bmatrix}$$

$$\tilde{A}_L = O_L A (O_L^T O_L)^{-1} O_L^T$$

$$\tilde{B}_L = \begin{bmatrix} O_L B & H_L^d \end{bmatrix} - \tilde{A}_L \begin{bmatrix} H_L^d & 0 \end{bmatrix}$$

2b) Projections

$$\underbrace{Y_{0|L} U_{0|L+g}^{\perp}}_{Z_{0|L}} = O_L \tilde{X}_0$$

where $Z_{0|L} = Y_{0|L} U_{0|L+g}^{\perp}$

and $U_x^{\perp} = I_{k \times k} - U_x^T (U_x U_x^T)^{+} U_x$ $\left\{ \begin{array}{l} x = 0/L+g \\ \text{for} \\ \text{simplicity} \\ \text{of notation} \end{array} \right.$

is such that $U \cdot U^{\perp} = 0$

$UU^{\perp} = U(I - U^T (UU^T)^{+} U) = U - U = 0$ qed

- $Z_{1|L} = Y_{1|L} U_{0|L+g}^{\perp}$

c) • Solve A from

$$Z_{1|L} = \underbrace{O_L A O_L^{+}}_{\tilde{A}_L} \cdot Z_{0|L}$$

- B and E solved from $\tilde{B}_L$:

$$\tilde{B}_L = (Y_{1|L} - \tilde{A}_L Y_{0|L}) U_{0|L+g}^{+}$$

$$= [\, x \;\; \begin{matrix} 0 \\ E \end{matrix} \,] \rightarrow E$$

d)

- In a first step we identify the innovations noise process $[\varepsilon_J\ \varepsilon_{J+1} \cdots \varepsilon_{N-1}] = \mathcal{E}$ by projecting ~~fut~~ past data $\begin{bmatrix} U_{0|J} \\ Y_{0|J} \end{bmatrix}$ onto future outputs in $Y_{J|1}$

We have

$$\underbrace{Y_{J|1} \Big/ \begin{bmatrix} U_{0|J} \\ Y_{0|J} \end{bmatrix}}_{Z^d_{J|1}} = D X_{J|1} \quad \Big\} \text{ Signal part}$$

when $A/B \overset{\text{def}}{=} A B^T (B B^T)^+$

$$Z^s_{J|1} = Y_{J|1} - Y_{J|1} \Big/ \begin{bmatrix} U_{0|J} \\ Y_{0|J} \end{bmatrix} \quad \Big\} \text{ Noise, innovations part.}$$

- When the noise part is known we solve a deterministic subspace identification problem

$$\left.\begin{aligned} x_{n+1} &= A x_n + \hat{B} \hat{u}_n \\ \hat{y}_n &= D x_n \end{aligned}\right\} \quad \left.\begin{matrix} \hat{u}_n \\ \hat{y}_n \end{matrix}\right\} \forall k = J, \dots N-1 \text{ known}$$