# Final Exam SCE2206
# System identification and optimal estimation
# Thursday June 8, 2016
# Time: kl. 9.00 - 14.00

The final exam consists of: 5 tasks.
The exam counts 100% of the final grade.
Available aids: pen and paper
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# Task 1 (20%): Realization theory, Kalman filter,Prediction error Method

$$length(0:h:t_f) \tag{1}$$
$$\mathbf{length}(0:h:t_f) \tag{2}$$

a) Assume known impulse responses

$$H_k = DA^{k-1}B \ \forall\ k = 1, \cdots, 6. \tag{3}$$

Answer the following:

- Write down the Hankel matrices $\mathbf{H}_{1|L}$ and $\mathbf{H}_{2|L}$ where you should use $L = 2$.
- How are $\mathbf{H}_{1|L}$ and $\mathbf{H}_{2|L}$ related to $O_L$, $C_J$ and $A$ ? (Here $O_L$ and $C_J$ are the extended observability and controllability matrices, respectively.)
- Show how you can find the system order, $n$, the extended observability matrix $O_L$ and the extended controllability matrix $C_J$ from a Singular Value decomposition (SVD).
- Find a formula for calculating the system matrix $A$ ?

b) Given a system modelled by a discrete time, state space model as follows

$$x_{k+1} = Ax_k + Bu_k + v_k, \tag{4}$$
$$y_k = Dx_k + Eu_k + w_k, \tag{5}$$

where $v_k$ is white process noise and $w_k$ is white measurements noise. Assume that the noise are uncorrelated, i.e. $\mathrm{E}(v_k w_k^T) = 0$.

- Write down a Kalman filter on apriori-aposteriori form for optimal estimation of the state vector $x_k$.
- Find a formula for the Kalman filter gain matrix, $K$, in the Kalman filter on apriori-aposteriori form.
- Show how the apriori-aposteriori formulation of the Kalman filter can be written as a Kalman filter on innovations form.

c) Given a non-linear system

$$x_{k+1} = f(x_k, u_k) + v_k \tag{6}$$
$$y_k = g(x_k) + w_k \tag{7}$$

where $v_k$ and $w_k$ are discrete white process noise and discrete white measurements noise, respectively.

Formulate the Kalman filter on apriori-aposteriori form for the non-linear system model in Eqs. (6) and (7).

d) The Kalman-filter on prediction form is often used in connection with Prediction error Methods (PEM) for system identification.

- Write down the Kalman filter on prediction form for the system in Eqs. (4) and (5) ?
- Define the Prediction Error (PE) $\varepsilon_k$ ?
- Assume that we want to use PEM to find the parameters in a single input single output system with $n = 2$ states. Write down the structure of a 2nd order model on canonical form and define the parameter vector $\theta$ ?

# Task 2 (20%): Autonomous systems

Assume given measured outputs

$$y_k \ \forall \ k = 0, \ldots, N-1 \tag{8}$$

from an autonomous system described by the model

$$\begin{aligned} x_{k+1} &= Ax_k, \ (x_0 \neq 0 \text{ The initial state}) \quad (9) \\ y_k &= Dx_k, \quad (10) \end{aligned}$$

where $x_k \in \mathbb{R}^n$ is the state vector and $x_0$ is the initial state vector at the initial time instant $k = 0$.
The **problem** in this task is to identify the system order $n$, the model matrices $A$ and $D$ as well as the initial state vector $x_0$ !

a) Matrix equations:

- Write down one matrix equation involving the Hankel matrix $Y_{0|L}$, the extended Observability matrix $O_L$ and a matrix $X_0$ with states.
- Write down one matrix equation involving the Hankel matrix $Y_{1|L}$, the extended Observability matrix $O_L$, a matrix $X_0$ with states and the system matrix $A$.
- Define the structure of the matrices $Y_{0|L}$, $Y_{1|L}$ and $X_0$.
- What is the importance of the matrix $A$ in eq. (9) ? Explain some properties !

b) Describe how $n$, $O_L$ and $X_0$ may be estimated from a Singular Value Decomposition (SVD) of one of the Hankel matrices $Y_{0|L}$ or $Y_{1|L}$ ?

c) Find a formula for estimating/calculating the system matrix $A$ ?

d) How can we find estimates of the output matrix $D$ and the initial state vector $x_0$ ?

# Task 3 (25%): Least Squares Methods

a) Consider a general ARX model

$$A(q)y_k = B(q)u_k + e_k, \tag{11}$$

where $e_k$ is white noise with covariance matrix $\Delta = \mathrm{E}(e_k e_k^T)$. $e_k$ is assumed uncorrelated with $u_k$. $A(q)$ and $B(q)$ are polynomials in the shift operator $q^{-1}$ such that e.g. $q^{-1}y_k = y_{k-1}$ and given by

$$A(q) = 1 + a_1 q^{-1} + a_2 q^{-2} + \cdots + a_{n_a} q^{-n_a}, \tag{12}$$
$$B(q) = b_0 + b_1 q^{-1} + b_2 q^{-2} + \cdots + b_{n_b} q^{-n_b}, \tag{13}$$

and where $n_a$ and $n_b$ are the order of the polynomials.

- Write the ARX model as a linear regression model of the form

$$y_k = \varphi_k^T \theta_0 + e_k \tag{14}$$

  In particular, define the regression vector, $\varphi_k$, and the parameter vector, $\theta_0$.

- Based on the regression model in Eq. (14) above, find a predictor, $\bar{y}_k(\theta)$, for the measurement $y_k$.
- Define the prediction error, $\varepsilon_k$.

b) Consider the following prediction error criterion

$$V_N(\theta) = \frac{1}{N}\sum_{k=1}^{N} \varepsilon_k^T \Lambda \varepsilon_k \tag{15}$$

where $\Lambda$ is a specified and symmetric weighting matrix.

- Find the Ordinary Least Squares (OLS) estimate, $\hat{\theta}_N$, of the true parameter vector $\theta_0$.
- Does there exist an optimal weighting matrix $\Lambda$ ? If so, what is the name of the corresponding estimate and the optimal $\Lambda$?

c) Assume given $t$ observations of a variable $y_k$, say

$$y_k \ \forall \ k = 1, \ldots, t \tag{16}$$

The mean after $t$ observations is given by

$$\bar{y}_t = \frac{1}{t}\sum_{k=1}^{t} y_k. \tag{17}$$

Find a recursive formula for calculating the mean ?

d) Assume that the observations $y_k$ as given in (19) above can be expressed as

$$y_k = \theta + e_k, \tag{18}$$

where $\theta$ is a constant parameter and $e_k$ is a white noise disturbance. Find a recursive algorithm/formula for calculating an estimate $\hat{\theta}$ of $\theta$ ?

e) Based on the OLS solution in Task 2b) above, show how we can develop a recursive Ordinary Least Squares (ROLS) method of the following form

$$\hat{\theta}_t = \hat{\theta}_{t-1} + K_t(y_t - \varphi_t^T \hat{\theta}_{t-1}). \tag{19}$$

You shall in particular find equations for computing the gain, $K_t$, in Equation (19).

# Task 4 (20%): Subspace System Identification: Deterministic and combined systems with feedback in data

Consider the discrete time deterministic model, ie.

$$\begin{aligned} x_{k+1} &= Ax_k + Bu_k, && (20) \\ y_k &= Dx_k + Eu_k, && (21) \end{aligned}$$

where the following output and input data matrices are known

$$Y = \begin{bmatrix} y_0^T \\ y_1^T \\ \vdots \\ y_{N-1}^T \end{bmatrix} \in \mathbb{R}^{N\times m}, \; U = \begin{bmatrix} u_0^T \\ u_1^T \\ \vdots \\ u_{N-1}^T \end{bmatrix} \in \mathbb{R}^{N\times r}. \tag{22}$$

a) Based on the model in Equations (20) and (21) and with known data as given in (22) we can develop the following matrix equations

$$\begin{aligned} Y_{0|L} &= O_L X_0 + H_L^d U_{0|L+g-1}, && (23) \\ Y_{1|L} &= \tilde{A}_L Y_{0|L} + \tilde{B}_L U_{0|L+g}, && (24) \end{aligned}$$

where $L \geq 1$ is a user specified positive integer.

- Write down the structure of the matrices in the matrix equations, (23) and (24), with parameters $N = 10$, $L = 2$, $J = 2$ and $g = 1$.
- Write down the expressions for the matrices $\tilde{A}_L$ and $\tilde{B}_L$ !

b) By using (22) and Equations (23) and (24) we may formulate the equations

$$Z_{0|L} = O_L X_0^a \tag{25}$$

and

$$Z_{1|L} = \tilde{A}_L Z_{0|L} \tag{26}$$

Find expressions for the data matrices $Z_{0|L}$ and $Z_{1|L}$.

Remark: define the projections which is involved in the expressions for $Z_{1|L}$ and $Z_{0|L}$.

c)

- Show how the system matrix $A$ can be estimated ?
- Describe shortly how the matrices $B$ and $E$ may be estimated ?

d) Consider a combined deterministic and stochastic system described by the following model on innovations form, ie.

$$\begin{aligned} x_{k+1} &= Ax_k + Bu_k + Ce_k, & (27) \\ y_k &= Dx_k + Fe_k & (28) \end{aligned}$$

where $e_k$ is white noise with unit covariance matrix, i.e., $\mathrm{E}(e_k e_k^T) = I$ and where the output and input data matrices as in Eq. (22) are known

Consider that the known input and output data as given in (22) are collected in closed loop, i.e., we assume that there is feedback in the known data.

From Eq. (28) we may define the matrix Equation

$$Y_{J|1} = DX_{J|1} + FE_{J|1} \tag{29}$$

- Define the Hankel matrices $Y_{J|1}$, $X_{J|1}$ and $E_{J|1}$.
  When $J \to \infty$ we can prove that the following identities holds

$$X_{J|1} = X_{J|1} / \begin{bmatrix} U_{0|J} \\ Y_{0|J} \end{bmatrix} \tag{30}$$

and

$$E_{J|1} / \begin{bmatrix} U_{0|J} \\ Y_{0|J} \end{bmatrix} = 0. \tag{31}$$

Use (30), (31) and (29) to find a projection

$$Z^s_{J|1} = FE_{J|1} \tag{32}$$

such that the innovations sequence

$$\begin{aligned} Z^s_{J|1} &= \begin{bmatrix} Fe_J & Fe_{J+1} & \dots & Fe_{N-1} \end{bmatrix} \\ &= \begin{bmatrix} \varepsilon_J & \varepsilon_{J+1} & \dots & \varepsilon_{N-1} \end{bmatrix} \end{aligned} \tag{33}$$

is estimated and hence could be considered as known.

- Explain how this projection can be used in order to develop a subspace identification algorithm for closed loop systems?

# Task 5 (15%): Diverse questions

a) Given a 1st order system (ie. a system with one state)

$$x_{k+1} = \theta_1 x_k + \theta_2 u_k + \theta_1 e_k, \tag{34}$$

$$y_k = x_k + e_k, \tag{35}$$

where $\theta_1$ and $\theta_2$ are unknown parameters and $e_k$ is white noise.

Find a linear regression model of the form

$$y_k = \varphi_k^T \theta + e_k. \tag{36}$$

Define the parameter vector $\theta$ and the vector $\varphi_k$ of regressors.

b) Consider there is a relationship between a variable $y_k$ and the time $t_k$ where $k = 1, \dots, N$ is discrete time.

Assume the following polynomial relationship between $y_k$ and $t_k$

$$y_k = a_0 + a_1 t_k + a_2 t_k^2 + e_k \tag{37}$$

where $a_0$, $a_1$ and $a_2$ are constant parameters and $e_k$ is some equation error.

- Find a linear regression model of the form

$$y_k = \varphi_k^T \theta + e_k. \tag{38}$$

Define the parameter vector $\theta$ and the vector $\varphi_k$ of regressors.

- From the known data $(y_k, t_k)$ formulate a linear matrix model

$$Y = XB + E. \tag{39}$$

Define the matrices $Y$, $X$, $E$ and $B$.

- Find an expression of the OLS estimate $B_{OLS}$ of $B$ ?

c) Orthogonal projections. Consider the linear equation

$$Y = OX + E, \tag{40}$$

where the matrices $Y \in \mathbb{R}^{m\times N}$ and $X \in \mathbb{R}^{n\times N}$ are known matrices. The matrix $E \in \mathbb{R}^{m\times N}$ is a matrix of noise and uncorrelated with $X$. We assume $N > n$.

- Define the mathematical definitions for the orthogonal projections $Y/X$ and $YX^{\perp}$ ?
- Write down a two dimensional figure to illustrate the projections?
- Is $Y = Y/X + YX^{\perp}$ correct or not ?