Solution proposal
Final exam SCE2206 2/6-15

Task 4

a)

- $Y_{1|L} = O_L X_0$

  $Y_{1|L} = O_L A X_0$

where

$$Y_{1|L} = \begin{bmatrix} y_1 & y_2 & \cdots & x \\ y_2 & y_3 & \cdots & x \\ \vdots & & & \\ y_L & y_{L+1} & \cdots & y_{N-1} \end{bmatrix}$$

$$Y_{0|L} = \begin{bmatrix} y_0 & y_1 & \cdots & x \\ y_1 & y_2 & \cdots & x \\ \vdots & \vdots & & \\ y_{L-1} & y_L & \cdots & y_{N-2} \end{bmatrix}$$

$$X_0 = [x_0 \; x_1 \; \cdots \;]$$

The number of columns in the Hankel matrices is

$$K = N - L$$

- A is the transition matrix
  - Given a continuous system $\dot{x}_c = A_c x_c$ then $A = e^{A_c \Delta t}$ where Δt sampling time
  - The stability of the system may be analyzed from the eigenvalues of A, inside unit circle for stability
  - + a number of properties, - $A \in \mathbb{R}^{n \times n}$

b) Take the SVD

$$Y_{0|L} = USV^T = [U_1\ U_2]\begin{bmatrix} S_1 & 0 \\ 0 & S_2 \end{bmatrix}[V_1\ V_2]^T$$
$$= U_1 S_1 V_1^T + U_2 S_2 V_2^T$$

Since the system is deterministic, no noise and assuming observability, $S_2 \approx 0$ and

$$Y_{0|L} \approx U_1 S_1 V_1^T$$

S_1 have n singular values $s_1 \geq s_2 \geq \cdots \geq s_n > 0$ on the diagonal and n the system order

We may take the output normal realization

$$O_L = U_1 \quad , \quad X_0 = S_1 V_1^T$$

$n = \text{rank}(Y_{0|L}) = \#$ of non-zero singular values of $Y_{0|L}$

c) We may find A from

$$Y_{1|L} = O_L A X_0 = U_1 A S_1 V_1^T$$

Solve for A.

$$U_1^T Y_{1|L} V_1 = A S_1$$

$$\Downarrow$$

$$\underline{\underline{A = U_1^T Y_{1|L} V_1 S_1^{-1}}}$$

d) • $D = U_1(1:m, :)$ (upper part of $O_2 = U_1$)

• $x_0 = X_0(:,1)$ when $X_0 = S_1 V_1^T$, ie first column of $X_0 = S_1 V_1^T$.

Task 2

a)

• $$y_k = \overbrace{\left[-y_{k-1}\ -y_{k-2}\ \cdots\ -y_{k-na}\ u_k\ u_{k-1}\ u_{k-2} \cdots u_{k-nb}\right]}^{\varphi_k^T} \overbrace{\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{na} \\ b_0 \\ b_1 \\ b_2 \\ \vdots \\ b_{nb} \end{bmatrix}}^{\theta}$$

# of parameters is $p = na + nb + 1$

• A predictor is

$$\bar{y}_k = \varphi_k^T \bar{\theta}_k$$

• The prediction error

$$\varepsilon_k = y_k - \bar{y}_k = y_k - \varphi_k^T \bar{\theta}_k$$

when $\bar{\theta}_k$ is the predicted parameter vector at time k.

2b)

$$\hat{\theta}_N = \left(\sum_{k=1}^{N} \varphi_k \Lambda \varphi_k^T \right)^{-1} \sum_{k=1}^{N} \varphi_k \Lambda y_k$$

- The optimal weighting matrix

$$\Lambda = \Delta^{-1}$$

where Δ is the covariance matrix of the equation error (noise) e_k

$$\Delta = E(e_k e_k^T) \triangleq \frac{1}{N} \sum_{k=1}^{N} e_k e_k^T$$

where N "large".

c) $$\bar{y}_t = \bar{y}_{t-1} + \frac{1}{t}(y_t - \bar{y}_{t-1})$$

with initial "guess" $\bar{y}_0$ specified

d) $$\hat{\theta}_t = \hat{\theta}_{t-1} + \frac{1}{t}(y_t - \hat{\theta}_{t-1})$$

with initial guess $\hat{\theta}_0$ specified

$$\hat{\theta}_N = \frac{1}{N} \sum_{k=1}^{N} y_k \quad \text{or} \quad \hat{\theta}_t = \frac{1}{t} \sum_{k=1}^{t} y_k$$

2e)

$$\hat{\theta}_t = \hat{\theta}_{t-1} + K_t \left(y_t - \varphi_t^T \theta_{t-1} \right)$$

$$K_t = P_t \varphi_t \Lambda$$

$$P_t = P_{t-1}^{-1} + \varphi_t \Lambda \varphi_t^T$$

Task 3

a)

$$\bar{y}_k = D \bar{x}_k + E u_k$$

$$\hat{x}_k = \bar{x}_k + K_k \overbrace{\left(y_k - \bar{y}_k \right)}^{\varepsilon_k}$$

$$\bar{x}_{k+1} = A \hat{x}_k + B u_k$$

$$K_k = \bar{X}_k D^T \left(D \bar{X}_k D^T + W \right)^{-1}$$

$$\hat{X}_k = (I - K_k D_k) \bar{X}_k (I - K_k D_k)^T + K_k W K_k^T$$

$$\bar{X}_{k+1} = A \hat{X} A^T + V$$

If stationary filter, this gives $\bar{X}$ as a solution to the Riccati eq.!

b)

$$\bar{y}_k = g(\bar{x}_k) \quad \text{Predicted output}$$

$$\hat{x}_k = \bar{x}_k + K_k \overbrace{(y_k - \bar{y}_k)}^{e\ \varepsilon_k}$$

$$\bar{x}_{k+1} = f(\bar{x}_k, u_k)$$

If time varying K, eqs as in 3a) but with

$$A_k = \left.\frac{df}{dx_k^T}\right|_{\hat{x}_k}, \quad D_k = \left.\frac{dg}{dx_k^T}\right|_{\bar{x}_k}$$

Task 4

a) With parameters $N=10$, $L=2$, $g=0$

Note: $y=2$ not needed!

$$Y_{1|L} = Y_{1|2} = \begin{bmatrix} y_1 & y_2 & y_3 & y_4 & y_5 & y_6 & y_7 & y_8 \\ y_2 & y_3 & y_4 & y_5 & y_6 & y_7 & y_8 & y_9 \end{bmatrix}$$

Note $K = N-2 = 8$ colums in all the Hankel matricer

$$U_{0|L+g} = U_{0|2} = \begin{bmatrix} u_0 & u_1 & u_2 & \cdots & & & & u_7 \\ u_1 & u_2 & u_3 & u_4 & u_5 & u_6 & u_7 & u_8 \end{bmatrix}$$

$$X_0 = \begin{bmatrix} x_0 & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 \end{bmatrix}$$

$$O_2 = \begin{bmatrix} D \\ DA \end{bmatrix}, \quad H_L^d = H_2^d = \begin{bmatrix} 0 \\ DB \end{bmatrix}$$

$$\tilde{A}_L = O_L A (O_L^T O_2)^{-1} O_L^T$$

$$\tilde{B}_2 = [O_L B \;\; H_L^d] - \hat{A}_2 [H_2^d \;\; 0]$$

Define the orthogonal projection matrix

$$U_{0|L+g}^{\perp} = I_k - U_{0|L+g}^T \left(U_{0|L+g} U_{0|L+g}^T\right)^{+} U_{0|L+g}$$

Then

$$Z_{0|L} = O_L X_0^a = Y_{0|L} U_{0|L+g}^{\perp} \qquad (1)$$

$$Z_{1|L} = \hat{A}_L Z_{0|L} = Y_{1|L} U_{0|L+g}^{\perp} \qquad (2)$$

c) Estimate n, O_L and X_0^a from SVD

$$Z_{0|L} = O_L X_0^a \approx U_1 S_1 V_1^T$$

Take $O_L = U_1$ and $X_0^a = S_1 V_1^T$ (not needed)

Solve A from (2) using $O_L = U_1$

$Z_{1|L} = U_1 A S_1 V_1^T$ since $(U_1^T U_1)^{-1} U_1^T \cdot U_1 = I$

$U_1^T Z_{1|L} V_1 = A S_1$

$$\Rightarrow \underline{A = U_1^T Z_{1|L} V_1 S_1^{-1}}$$

d) YES

Task 5

a)

$$Z^{s}_{y|1} = F E_{y|1} = \mathcal{E}_{y|1} = Y_{y|1} - Y_{y|1} / \begin{bmatrix} U_{0|J} \\ Y_{0|J} \end{bmatrix}$$

Hence, the innovations noise

$\mathcal{E}_k \ \forall \ k = J, J+1, \ldots, N-1$

is known and we may solve a deterministic id. problem as in task 4 in a 2nd step.

b) $A/B = AB^T (BB^T)^+ B$

when $(\cdot)^+ = (\cdot)^{-1}$ if it exist

$(\cdot)^+$ - pseudo inverse