

Final Exam SCE2206
**System identification and optimal
estimation**

Thursday June 2, 2015

Time: kl. 9.00 - 14.00

The final exam consists of: 5 tasks.
The exam counts 80% of the final grade.

Available aids: pen and paper

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Task 1 (20%): Autonomous systems

Assume given measured outputs

$$y_k \quad \forall k = 0, \dots, N - 1 \quad (1)$$

from an autonomous system described by the model

$$x_{k+1} = Ax_k, \quad (x_0 \neq 0 \text{ The initial state}) \quad (2)$$

$$y_k = Dx_k, \quad (3)$$

where $x_k \in \mathbb{R}^n$ is the state vector and x_0 is the initial state vector at the initial time instant $k = 0$.

The **problem** in this task is to identify the system order n , the model matrices A and D as well as the initial state vector x_0 !

a) Matrix equations:

- Write up one matrix equation involving the Hankel matrix $Y_{0|L}$, the extended Observability matrix O_L and a matrix X_0 with states.
- Write up one matrix equation involving the Hankel matrix $Y_{1|L}$, the extended Observability matrix O_L , a matrix X_0 with states and the system matrix A .
- Define the structure of the matrices $Y_{0|L}$, $Y_{1|L}$ and X_0 .
- What is the importance of the matrix A in eq. (2) ? Explain some properties !

b) Describe how n , O_L and X_0 may be estimated from a Singular Value Decomposition (SVD) of the Hankel matrix $Y_{0|L}$.

c) Find a formula for estimating/calculating the system matrix A ?

d) How can we find estimates of the output matrix D and the initial state vector x_0 ?

Task 2 (25%): Least Squares methods

a) Consider a general ARX model

$$A(q)y_k = B(q)u_k + e_k, \quad (4)$$

where e_k is white noise with covariance matrix $\Delta = E(e_k e_k^T)$. e_k is assumed uncorrelated with u_k . $A(q)$ and $B(q)$ are polynomials in the shift operator q^{-1} such that e.g. $q^{-1}y_k = y_{k-1}$ and given by

$$A(q) = 1 + a_1 q^{-1} + a_2 q^{-2} + \dots + a_{n_a} q^{-n_a}, \quad (5)$$

$$B(q) = b_0 + b_1 q^{-1} + b_2 q^{-2} + \dots + b_{n_b} q^{-n_b}, \quad (6)$$

and where n_a and n_b are the order of the polynomials.

- Write the ARX model as a linear regression model of the form

$$y_k = \varphi_k^T \theta_0 + e_k \quad (7)$$

In particular, define the regression vector, φ_k , and the parameter vector, θ_0 .

- Based on the regression model in Eq. (7) above, find a predictor, $\bar{y}_k(\theta)$, for the measurement y_k .
- Define the prediction error, ε_k .

b) Consider the following prediction error criterion

$$V_N(\theta) = \frac{1}{N} \sum_{k=1}^N \varepsilon_k^T \Lambda \varepsilon_k \quad (8)$$

where Λ is a specified and symmetric weighting matrix.

- Find the Ordinary Least Squares (OLS) estimate, $\hat{\theta}_N$, of the true parameter vector θ_0 .
- Does there exist an optimal weighting matrix Λ ? If so, what is the name of the corresponding estimate and the optimal Λ ?

c) Assume given t observations of a variable y_k , say

$$y_k \quad \forall k = 1, \dots, t \quad (9)$$

The mean after t observations is given by

$$\bar{y}_t = \frac{1}{t} \sum_{k=1}^t y_k. \quad (10)$$

Find a recursive formula for calculating the mean ?

- d) Assume that the observations y_k as given in (12) above can be expressed as

$$y_k = \theta + e_k, \quad (11)$$

where θ is a constant parameter and e_k is a white noise disturbance.

Find a recursive algorithm/formula for calculating an estimate $\hat{\theta}$ of θ ?

- e) Based on the OLS solution in Task 2b) above, show how we can develop a recursive Ordinary Least Squares (ROLS) method of the following form

$$\hat{\theta}_t = \hat{\theta}_{t-1} + K_t(y_t - \varphi_t^T \hat{\theta}_{t-1}). \quad (12)$$

You shall in particular find equations for computing the gain, K_t , in Equation (12).

Task 3 (10%): The Kalman filter

- a) Given a system modeled by a discrete time, state space model as follows

$$x_{k+1} = Ax_k + Bu_k + v_k, \quad (13)$$

$$y_k = Dx_k + Eu_k + w_k, \quad (14)$$

where v_k is white process noise and w_k is white measurements noise. Assume that the noise are uncorrelated, i.e. $E(v_k w_k^T) = 0$.

- Write down a Kalman filter on apriori-aposteriori form for optimal estimation of the state vector x_k .
- Find a formula for the Kalman filter gain matrix, K , in the Kalman filter on apriori-aposteriori form.
- Show how the apriori-aposteriori formulation of the Kalman filter can be written as a Kalman filter on innovations form.
- Write down the Kalman-filter on prediction form for the system in Eqs. (13) and (14) ? Remark: The Kalman filter on prediction form is often used to calculate a prediction \bar{y}_k of the output y_k and used in prediction error methods for system identification.

- b) Given a non-linear system

$$x_{k+1} = f(x_k, u_k) + v_k, \quad (15)$$

$$y_k = g(x_k) + w_k, \quad (16)$$

where v_k and w_k are discrete white process noise and discrete white measurements noise, respectively.

Formulate the Kalman filter on apriori-aposteriori form for the non-linear system model in Eqs. (15) and (16).

Task 4 (15%): Deterministic Subspace System Identification

Consider the discrete time deterministic model, ie.

$$x_{k+1} = Ax_k + Bu_k, \quad (17)$$

$$y_k = Dx_k + Eu_k, \quad (18)$$

where the following output and input data matrices are known

$$Y = \begin{bmatrix} y_0^T \\ y_1^T \\ \vdots \\ y_{N-1}^T \end{bmatrix} \in \mathbb{R}^{N \times m}, \quad U = \begin{bmatrix} u_0^T \\ u_1^T \\ \vdots \\ u_{N-1}^T \end{bmatrix} \in \mathbb{R}^{N \times r}. \quad (19)$$

- a) Based on the model in Equations (17) and (18) and with known data as given in (19) we can develop the following matrix equations

$$Y_{0|L} = O_L X_0 + H_L^d U_{0|L+g-1}, \quad (20)$$

$$Y_{1|L} = \tilde{A}_L Y_{0|L} + \tilde{B}_L U_{0|L+g}, \quad (21)$$

where $L \geq 1$ is a user specified positive integer.

- Write down the structure of the matrices in the matrix equations, (20) and (21), with parameters $N = 10$, $L = 2$, $J = 2$ and $g = 0$.
 - Write up the expressions for the matrices \tilde{A}_L and \tilde{B}_L !
- b) By using (19) and Equations (20) and (21) we may formulate the equations

$$Z_{0|L} = O_L X_0^a \quad (22)$$

and

$$Z_{1|L} = \tilde{A}_L Z_{0|L} \quad (23)$$

Find expressions for the data matrices $Z_{0|L}$ and $Z_{1|L}$.

Remark: define the projections which is involved in the expressions for $Z_{1|L}$ and $Z_{0|L}$.

- c) Show how

- the system matrix A

can be estimated.

- d) Assume that the system is single output. Is it then possible to write the deterministic system as a linear regression model? , i.e., as a model

$$y_k = \varphi_k^T \theta \quad (24)$$

where φ_k contains the regressors and θ is a vector of parameters. The answer is YES? or NO?

Task 5 (10%): Subspace System Identification: Combined Systems with feedback in the data

Consider the discrete time model on innovations form, ie.

$$x_{k+1} = Ax_k + Bu_k + Ce_k, \quad (25)$$

$$y_k = Dx_k + Fe_k \quad (26)$$

where e_k is white noise with unit covariance matrix, i.e., $E(e_k e_k^T) = I$ and where the following output and input data matrices are known

$$Y = \begin{bmatrix} y_0^T \\ y_1^T \\ \vdots \\ y_{N-1}^T \end{bmatrix} \in \mathbb{R}^{N \times m}, \quad U = \begin{bmatrix} u_0^T \\ u_1^T \\ \vdots \\ u_{N-1}^T \end{bmatrix} \in \mathbb{R}^{N \times r}. \quad (27)$$

Consider that the known input and output data as given in (27) are collected in closed loop, i.e., we assume that there is feedback in the known data.

a)

From Eq. (26) we may define the matrix Equation

$$Y_{J|1} = DX_{J|1} + FE_{J|1} \quad (28)$$

- Define the Hankel matrices $Y_{J|1}$, $X_{J|1}$ and $E_{J|1}$.
- When $J \rightarrow \infty$ we can prove that the following identities holds

$$X_{J|1} = X_{J|1} / \begin{bmatrix} U_{0|J} \\ Y_{0|J} \end{bmatrix} \quad (29)$$

and

$$E_{J|1} / \begin{bmatrix} U_{0|J} \\ Y_{0|J} \end{bmatrix} = 0. \quad (30)$$

Use (29), (30) and (28) to find a projection

$$Z_{J|1}^s = FE_{J|1} \quad (31)$$

such that the innovations sequence

$$\begin{aligned} Z_{J|1}^s &= [Fe_J \ Fe_{J+1} \ \dots \ Fe_{N-1}] \\ &= [\varepsilon_J \ \varepsilon_{J+1} \ \dots \ \varepsilon_{N-1}] \end{aligned} \quad (32)$$

is estimated and hence could be considered as known.

- Explain how this projection can be used in order to develop a subspace identification algorithm for closed loop systems.
- b) Given two matrices A and B . Define the mathematical expression for the orthogonal projection of A onto B , i.e. define

$$A/B = ? \quad (33)$$