SCE2206 System Identification 4/6-14

Final exam 2014, 5/6-14 Solution proposal

Task 1

a) $y_{k+1} - e_{k+1} = \theta_1 (y_k - e_k) + \theta_2 u_k + \theta_1 e_k$

$\Updownarrow$

$y_{k+1} = \theta_1 y_k + \theta_2 u_k + e_{k+1}$

$\Updownarrow$

$$y_k = \begin{bmatrix} y_{k-1} & u_{k-1} \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} + e_k = \varphi_k^T \theta + e_k$$

where $\varphi_k = \begin{bmatrix} y_{k-1} \\ u_{k-1} \end{bmatrix}$, $\theta = \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix}$

b)

- $$y_k = \begin{bmatrix} 1 & t_k & t_k^2 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} + e_k = \varphi_k^T \theta + e_k$$

where $\varphi_k^T = \begin{bmatrix} 1 & t_k & t_k^2 \end{bmatrix}$, $\theta = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix}$

- We have $Y = XB + E$, where

$$Y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{bmatrix}, \quad X = \begin{bmatrix} 1 & t_1 & t_1^2 \\ 1 & t_2 & t_2^2 \\ \vdots & \vdots & \vdots \\ 1 & t_N & t_N^2 \end{bmatrix}, \quad B = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix}, \quad E = \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_N \end{bmatrix},$$

- $B_{OLS} = (X^T X)^{-1} X^T Y$

c)

$$H_{2|3} = \begin{bmatrix} H_2 & H_3 & H_4 & H_5 & H_6 \\ H_3 & H_4 & H_5 & H_6 & H_7 \\ H_4 & H_5 & H_6 & H_7 & H_8 \end{bmatrix} = O_3 A C_5$$

Here $L = 3$ and $y = 5$

$$H_{1|3} = \begin{bmatrix} H_1 & H_2 & H_3 & H_4 & H_5 \\ H_2 & H_3 & H_4 & H_5 & H_6 \\ H_3 & H_4 & H_5 & H_6 & H_7 \end{bmatrix} = O_3 \cdot C_5$$

$$H_{1|L} = USV^T = [U_1\ U_2] \begin{bmatrix} S_1 & 0 \\ 0 & \underbrace{S_2}_{\approx 0} \end{bmatrix} [V_1\ V_2]^T \approx U_1 S_1 V_1^T$$

$n = \text{rank}(H_{1|L})$ and $n = \#$ of non-zero singular values in S and $S_1 \in \mathbb{R}^{n \times n}$.

$\underline{O_L = U_1 \text{ and } C_y = S_1 V_1^T}$

$\underline{D = O_L(1{:}m, :) \text{ and } B = C_y(:, 1{:}r)}$

$$O_L = \begin{bmatrix} D \\ DA \\ \vdots \\ DA^{L-1} \end{bmatrix}, \quad C_y = [B\ AB \cdots A^{y-1}B]$$

- Solve $H_{2|L} = H_{2|L} = O_L A C_y$ for A

$$\underline{A = (O_L^T O_L)^{-1} O_L^T H_{2|L} C_y^T (C_y C_y^T)^{-1}}$$

d) • The observability matrix of the identified model is O_L

$$O_L = \begin{bmatrix} D \\ DA \\ \vdots \\ DA^{L-1} \end{bmatrix}$$

• Transformed identified model is

$$x_{k+1}^{p} = T^{-1}AT \cdot x_p + T^{-1}Bu_k$$
$$y_k = DTx_k^{p}$$

Observability matrix of transformed model is

$$\begin{bmatrix} DT \\ DT \cdot T^{-1}AT \\ \vdots \\ DT \cdot T^{-1}A^{L-1}T \end{bmatrix} = O_L \cdot T$$

Observability matrix of physical model is

$$O_L^{p} = \begin{bmatrix} D_p \\ D_p A_p \\ \vdots \\ D_p A_p^{L-1} \end{bmatrix}$$

• Putting $O_L T = O_L^{p}$ gives T

$$\underline{\underline{T = (O_L^T O_L)^{-1} O_L^T O_L^{p}}}$$

Hence, the following transformed DSR model will have the same structure as the physical system

$$x_{k+1}^{p} = A_{dsr} x_k^{p} + B_{dsr} u_k$$

$$y_k = D_{dsr} x_k^{p}$$

where

$$A_{dsr} = T^{-1} A T \; , \; B_{dsr} = T^{-1} B$$

$$D_{dsr} = D T$$

where $[A, B, D, E, C, F, x_0] = dsr(Y, U, L)$ outputs (A, B, D) from dsr method and

$$T = (O_L^T O_L)^{-1} O_L^T O_L^{p}$$

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a) We write the ARX model as

$$A(q)y_k = B(q)u_k + e_k$$

and

$$y_k + a_1 y_{k-1} + a_2 y_{k-2} + \cdots + a_{n_a} y_{k-n_a} = b_1 u_{k-1} + b_2 u_{k-2} + \cdots + b_{n_b} u_{k-n_b} + e_k$$

$$\Updownarrow$$

$$y_k = \overbrace{\begin{bmatrix} -y_{k-1} & -y_{k-2} & \cdots & -y_{k-n_a} & u_{k-1} & u_{k-2} & \cdots & u_{k-n_b} \end{bmatrix}}^{\varphi_k^T} \overbrace{\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{n_a} \\ b_1 \\ b_2 \\ \vdots \\ b_{n_b} \end{bmatrix}}^{\theta} + e_k$$

$$= \varphi_k^T \theta + e_k$$

b)

$$\hat{\theta}_N = \left(\sum_{k=1}^{N} \varphi_k \Lambda \varphi_k^T \right)^{-1} \sum_{k=1}^{N} \varphi_k \Lambda y_k$$

Define the covariance matrix

$$\Delta = E(e_k e_k^T)$$

of the error term e_k.

Then $\Lambda = \Delta^{-1}$ is the optimal estimate, i.e. the BLUE estimator, Best Linear Unbiased Estimate!

3a)

$$\bar{y}_t = \bar{y}_{t-1} + \frac{1}{t}(y_t - \bar{y}_{t-1})$$

and

$$\bar{y}_0 = 0$$
$$\bar{y}_1 = y_1$$
$$\bar{y}_2 = \bar{y}_1 + \frac{1}{2}(y_2 - \bar{y}_1) = y_1 + \frac{1}{2}(y_2 - y_1)$$
$$\vdots$$

b) An estimate of θ equal to the mean

$$\hat{\theta}_t = \frac{1}{t}\sum_{k=1}^{t} y_k$$

Here we find

$$\frac{1}{t}\sum_{k=1}^{t} y_k = \frac{1}{t}\sum_{k=1}^{t}(\theta + e_k) = \frac{1}{t} t\theta + \frac{1}{t}\sum_{k=1}^{t} e_k = \theta$$

because e_k white zero mean

Same as in 3a)

$$\hat{\theta}_t = \hat{\theta}_{t-1} + \frac{1}{t}(y_t - \hat{\theta}_{t-1})$$

c)

$$\hat{\theta}_t = \hat{\theta}_{t-1} + K_t(y_t - \varphi_t^T \hat{\theta}_{t-1})$$

$$K_t = P_t \varphi_t \Lambda$$

$$P_t^{-1} = P_{t-1}^{-1} + \varphi_t \Lambda \varphi_t^T$$