Final Exam SCE2202
System identification and optimal estimation
Thursday June 5, 2014
Time: kl. 9.00 - 14.00

The final exam consists of: 5 tasks.
The exam counts 70% of the final grade.
Available aids: pen and paper
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Task 1 (20%): Diverse questions

a) Given a 1st order system (ie. a system with one state)

$$x_{k+1} = \theta_1 x_k + \theta_2 u_k + \theta_1 e_k, \tag{1}$$
$$y_k = x_k + e_k, \tag{2}$$

where θ_1 and θ_2 are unknown parameters and e_k is white noise.

Find a linear regression model of the form

$$y_k = \varphi_k^T \theta + e_k. \tag{3}$$

Define the parameter vector θ and the vector φ_k of regressors.

b) Consider there is a relationship between a variable y_k and the time t_k where $k = 1, \ldots, N$ is discrete time.

Assume the following polynomial relationship between y_k and t_k

$$y_k = a_0 + a_1 t_k + a_2 t_k^2 + e_k \tag{4}$$

where a_0, a_1 and a_2 are constant parameters and e_k is some equation error.

- Find a linear regression model of the form

$$y_k = \varphi_k^T \theta + e_k. \tag{5}$$

  Define the parameter vector θ and the vector φ_k of regressors.

- From the known data (y_k, t_k) formulate a linear matrix model

$$Y = XB + E. \tag{6}$$

  Define the matrices Y, X, E and B.

- Find an expression of the OLS estimate B_{OLS} of B ?

c) Assume known impulse responses

$$H_k = DA^{k-1}B \ \forall \ k = 1, \cdots, 8. \tag{7}$$

Answer the following:

- Write up the Hankel matrices $\mathbf{H}_{1|L}$ and $\mathbf{H}_{2|L}$ where you should use $L = 3$.
- Show how you can find the system order, n, the extended observability matrix O_L and the extended controllability matrix C_J from a Singular Value decomposition (SVD) of $\mathbf{H}_{1|L}$.

- How are $\mathbf{H}_{1|L}$ and $\mathbf{H}_{2|L}$ related to O_L and C_J ?
- Describe how the corresponding model matrices A, B and D may be identified/calculated !

d) Assume that we have identified a linear state space model of the form

$$x_{k+1} = Ax_k + Bu_k \tag{8}$$
$$y_k = Dx_k \tag{9}$$

from some known data matrices Y and U by some method, say the DSR method.

- Define the extended observability matrix O_L of the system in (8) and (9)?

Assume now that based on a non-linear model description of the true physical process we have obtained a known linearized state space model of the form

$$x^p_{k+1} = A_p x^p_k + B_p u_k \tag{10}$$
$$y_k = D_p x^p_k \tag{11}$$

where x^p_k represents the physical states and A_p, B_p and D_p are the model matrices in the physical model (10) and (11).

- Define the extended observability matrix O^p_L for the physical system/model (10) and (11)? How may we check that the state vector x^p_k are observable ?
- Assume the relationship $x_k = Tx^p_k$ between the state vector x_k in the identified model (8) and (9) and x^p_k in the physical model (10) and (11).

  How can we find the transformation matrix T ? Tips: Use that the observability matrix of the identified model and the physical model are equal.
- How can we transform the identified model (8) and (9) to an equivalent model with physical state representation, i.e. as a model represented with the physical state vector x^p_k ?

Task 2 (10%): Ordinary Least Squares method

a) Consider a general ARX process

$$y_k = \frac{B(q)}{A(q)} u_k + \frac{e_k}{A(q)} \tag{12}$$

where e_k is white noise uncorrelated with u_k. $A(q)$ and $B(q)$ are polynomials in the shift operator q^{-1} such that e.g. $q^{-1}y_k = y_{k-1}$ and given by

$$\begin{aligned} A(q) &= 1 + a_1 q^{-1} + a_2 q^{-2} + \cdots + a_{n_a} q^{-n_a}, & (13)\\ B(q) &= b_1 q^{-1} + b_2 q^{-2} + \cdots + b_{n_b} q^{-n_b}, & (14) \end{aligned}$$

and where n_a and n_b are the order of the polynomials.

- Write the ARX model as a linear regression model of the form

  $$y_k = \varphi_k^T \theta_0 + e_k \tag{15}$$

  In particular, define the regression vector, φ_k, and the parameter vector, θ_0.

- Based on the regression model in Eq. (15) above, find a predictor, $\bar{y}_k(\theta)$, for the measurement y_k.
- Define the prediction error, ε_k.

b) Consider the following prediction error criterion

$$V_N(\theta) = \frac{1}{N} \sum_{k=1}^{N} \varepsilon_k^T \Lambda \varepsilon_k \tag{16}$$

where Λ is a specified and symmetric weighting matrix.

- Find the Ordinary Least Squares (OLS) estimate, $\hat{\theta}_N$, of the true parameter vector θ_0.
- Does there exist an optimal weighting matrix Λ ? If so, what is the name of the corresponding estimate ?

Task 3 (15%): Recursive system identification

a) Assume given t observations of a variable, say $y_k \ \forall \ k = 1, \dots, t$. The mean after t observations is given by

$$\bar{y}_t = \frac{1}{t} \sum_{k=1}^{t} y_k. \tag{17}$$

Find a recursive formula for calculating the mean ?

b) Assume that the observations y_k as given in Task 3 a) above can be expressed as

$$y_k = \theta + e_k, \tag{18}$$

where θ is a constant and e_k is a disturbance.

Find a recursive algorithm/formula for calculating an estimate $\hat{\theta}$ of θ ?

c) Based on the OLS solution in Task 2 above, show how we can develop a recursive Ordinary Least Squares (ROLS) method of the following form

$$\hat{\theta}_t = \hat{\theta}_{t-1} + K_t(y_t - \varphi_t^T \hat{\theta}_{t-1}). \tag{19}$$

You shall in particular find equations for computing the gain, K_t, in Equation (19).

Task 4 (5%): The Kalman filter

a) Given a system modelled by a discrete time, state space model as follows

$$\begin{aligned} x_{k+1} &= Ax_k + Bu_k + v_k, & (20) \\ y_k &= Dx_k + Eu_k + w_k, & (21) \end{aligned}$$

where v_k is white process noise and w_k is white measurements noise. Assume that the noise are uncorrelated, i.e. $\mathrm{E}(v_k w_k^T) = 0$.

- Write down a Kalman filter on apriori-aposteriori form for optimal estimation of the state vector x_k.
- Find a formula for the Kalman filter gain matrix, K, in the Kalman filter on apriori-aposteriori form.
- Show how the apriori-aposteriori formulation of the Kalman filter can be written as a Kalman filter on innovations form.

- Write down the Kalman-filter on prediction form for the system in Eqs. (20) and (21) ? Remark: The Kalman filter on prediction form is often used to calculate a prediction $\bar{y}_k$ of the output y_k and used in prediction error methods for system identification.

b) Given a non-linear system

$$\begin{aligned} x_{k+1} &= f(x_k, u_k) + v_k, && (22) \\ y_k &= g(x_k) + w_k, && (23) \end{aligned}$$

where v_k and w_k are discrete white process noise and discrete white measurements noise, respectively.

Formulate the Kalman filter on apriori-aposteriori form for the non-linear system model in Eqs. (22) and (23).

Task 5 (15%): Subspace System Identification of Deterministic Systems: Using shift invariance principle

Consider the discrete time deterministic model, ie.

$$\begin{aligned} x_{k+1} &= Ax_k + Bu_k, && (24) \\ y_k &= Dx_k + Eu_k, && (25) \end{aligned}$$

where the following output and input data matrices are known

$$Y = \begin{bmatrix} y_0^T \\ y_1^T \\ \vdots \\ y_{N-1}^T \end{bmatrix} \in \mathbb{R}^{N \times m}, \; U = \begin{bmatrix} u_0^T \\ u_1^T \\ \vdots \\ u_{N-1}^T \end{bmatrix} \in \mathbb{R}^{N \times r}. \tag{26}$$

a) Based on the model in Eqs. (24) and (25) and with known data as given in (26) we can develop the following matrix equation

$$Y_{0|L+1} = O_{L+1} X_0 + H_{L+1}^d U_{0|L+g}, \tag{27}$$

where $L \geq 1$ is a user specified positive integer parameter.

Write down the structure of the matrices in the matrix Eq. (27), with parameters $N = 12$, $L = 2$ and $g = 0$.

b) By using the known input-output data (26) and Eq. (27) we may formulate the projected equation

$$Z_{0|L+1} = O_{L+1} X_0^a \tag{28}$$

Find expressions for the data matrix $Z_{0|L+1}$.

Remark: define the projections which is involved in the expressions for $Z_{0|L+1}$.

c) Show how

- the system order, n
- the extended observability matrix O_{L+1}
- the system matrices A and D

can be estimated.

Task 6 (5%): Subspace System Identification: Combined Systems with feedback in the data

Consider the discrete time model on innovations form, ie.

$$\begin{aligned} x_{k+1} &= Ax_k + Bu_k + Ce_k, \quad (29) \\ y_k &= Dx_k + Fe_k \quad (30) \end{aligned}$$

where e_k is white noise with unit covariance matrix, i.e., $\mathrm{E}(e_k e_k^T) = I$ and where the following output and input data matrices are known

$$Y = \begin{bmatrix} y_0^T \\ y_1^T \\ \vdots \\ y_{N-1} \end{bmatrix} \in \mathbb{R}^{N \times m}, \; U = \begin{bmatrix} u_0^T \\ u_1^T \\ \vdots \\ u_{N-1} \end{bmatrix} \in \mathbb{R}^{N \times r}. \tag{31}$$

Consider that the known input and output data as given in (31) are collected in closed loop, i.e., we assume that there is feedback in the known data.

a)

From Eq. (30) we may define the matrix Equation

$$Y_{J|1} = DX_{J|1} + FE_{J|1} \tag{32}$$

- Define the data matrices $Y_{J|1}$, $X_{J|1}$ and $E_{J|1}$.
- When $J \to \infty$ we can prove that the following identity holds

$$X_{J|1} = X_{J|1} / \begin{bmatrix} U_{0|J} \\ Y_{0|J} \end{bmatrix} \tag{33}$$

  Use (33) and (32) to find a projection

$$Z^s_{J|1} = F E_{J|1} \tag{34}$$

  such that the innovations sequence

$$\begin{aligned} Z^s_{J|1} &= \begin{bmatrix} F e_J & F e_{J+1} & \dots & F e_{N-1} \end{bmatrix} \\ &= \begin{bmatrix} \varepsilon_J & \varepsilon_{J+1} & \dots & \varepsilon_{N-1} \end{bmatrix} \end{aligned} \tag{35}$$

  could be identified.
- Explain how this projection can be used in order to develop a subspace identification algorithm for closed loop systems.