Exam 31/5 -12

1) We have

$$x^1_{k+1} = -a_1 x^1_k + x^2_k + b_1 u_k + k_1 e_k \qquad (1)$$

$$x^2_{k+1} = -a_2 x^1_k + b_2 u_k + k_2 e_k \qquad (2)$$

$$y_k = x^1_k + e_k \qquad (3)$$

From (1)

$$x^1_{k+2} = -a_1 x^1_{k+1} + x^2_{k+1} + b_1 u_{k+1} + k_1 e_{k+1}$$

Use x^2_{k+1} from (2) and we get

$$x^1_{k+2} = -a_1 x^1_{k+1} + (-a_2 x^1_k + b_2 u_k + k_2 e_k) + b_1 u_{k+1} + k_1 e_{k+1} \quad (4)$$

Use y_k from 3, i.e. substitute $x^1_k = y_k - e_k$ in (4)

$$y_{k+2} - e_{k+2} = -a_1(y_{k+1} - e_{k+1}) - a_2(y_k - e_k) + b_2 u_k + k_2 e_k + b_1 u_{k+1} + k_1 e_{k+1}$$

$$\Downarrow$$

$$y_{k+2} = \begin{bmatrix} -y_k & -y_{k+1} & u_k & u_{k+1} \end{bmatrix} \begin{bmatrix} a_2 \\ a_1 \\ b_2 \\ b_1 \end{bmatrix} + e_{k+2} + a_1 e_{k+1} + k_1 e_{k+1} + a_2 e_k + k_2 e_k$$

ARX model when

$$\underline{k_1 = -a_1} \quad \text{and} \quad \underline{k_2 = -a_2}$$

Then $y_k = \varphi_k^T \theta + e_k$

where

$$\underline{\varphi_k^T = \begin{bmatrix} -y_{k-2} & -y_{k-1} & u_{k-2} & u_{k-1} \end{bmatrix}}, \quad \underline{\theta = \begin{bmatrix} a_2 \\ a_1 \\ b_2 \\ b_1 \end{bmatrix}}$$

Notice that ~~W~~e have an

ARMAX model $\rightarrow e_{k+2}+(a_1+k_1)e_{k+1}+(a_2+k_2)e_k$

$$y_{k+2}+a_1 y_{k+1}+a_2 y_k = b_2 u_k + b_1 u_{k+1} + \cancel{e_{k+2}+k_1 e_{k+1}+k_2 e_k}$$

Using $q^{-1} y_k = y_{k-1}$ and $q\, y_{k-1} = y_k$

$$y_k + a_1 y_{k-1} + a_2 y_{k-2} = b_1 u_{k-1} + b_2 u_{k-2}$$

k_2+a_2

$$+\, e_k + k_1 e_{k-1} + k_2 e_{k-2}$$

k_1+a_1

and

$$\underbrace{\left(1+a_1 q^{-1}+a_2 q^{-2}\right)}_{A(q)} y_k = \overbrace{\left(b_1 q^{-1}+b_2 q^{-2}\right)}^{B(q)} u_k$$

k_2+a_2

$$+\underbrace{\left(1+k_1 q^{-1}+k_2 q^{-2}\right)}_{C(q)} e_k$$

k_1+a_1

This is an ARMAX model

$$A(q)\, y_k = B(q)\, u_k + C(q)\, e_k$$

where

$C(q)=1+(k_1+a_1)q^{-1}+(k_2+a_2)q^{-2}$

$B(q)=b_1 q^{-1}+b_2 q^{-2}$, $A(q)=1+a_1 q^{-1}+a_2 q^{-2}$ //

1e)

$y_k \;\forall\; k=1,\dots,t$

$$\hat{\theta}_t = \frac{1}{t}\sum_{k=1}^{t} y_k \qquad (1)$$

Divide up sum

$$\hat{\theta}_t = \frac{1}{t}\left(\sum_{k=1}^{t-1} y_k + y_t\right) \qquad (2)$$

From (1)

$$\hat{\theta}_{t-1} = \frac{1}{t-1}\sum_{k=1}^{t-1} y_k \Rightarrow \sum_{k=1}^{t-1} y_k = (t-1)\,\hat{\theta}_{t-1}$$

Put into (2)

$$\hat{\theta}_t = \frac{1}{t}\left((t-1)\hat{\theta}_{t-1} + y_t\right) =$$

$$= \frac{1}{t}\left(t\,\hat{\theta}_{t-1} + y_t - \hat{\theta}_{t-1}\right)$$

$$\boxed{\hat{\theta}_t = \hat{\theta}_{t-1} + \frac{1}{t}\left(y_t - \hat{\theta}_{t-1}\right)}$$