SCE2206
Solution exam 31/5-12

1

a) ARX model:

- $k_1 = -a_1$ and $k_2 = -a_2$

- May write

$$y_k = \varphi_k^T \theta_0 + e_k$$

where

$$\varphi_k^T = [-y_{k-2} \;\; -y_{k-1} \;\; u_{k-2} \;\; u_{k-1}]$$

$$\theta_0 = \begin{bmatrix} a_2 \\ a_1 \\ b_2 \\ b_1 \end{bmatrix}$$

or variants of this!

b)
- A predictor for y_k: $\bar{y}_k(\theta) = \varphi_k^T \theta$
- Prediction Error: $\varepsilon_k = y_k - \bar{y}_k(\theta)$

c)

$$\hat{\theta}_N = \left(\sum_{k=1}^{N} \varphi_k \Lambda \varphi_k^T \right)^{-1} \sum_{k=1}^{N} \varphi_k \Lambda y_k$$

d) $$\hat{\theta}_t = \hat{\theta}_{t-1} + K_t \left(y_t - \varphi_t \hat{\theta}_{t-1} \right.$$

where

$$P_t^{-1} = P_{t-1}^{-1} + \varphi_t \Lambda \varphi_t^T$$

$$K_t = P_t \varphi_t \Lambda$$

Algorithm

Step 1 Initialize, e.g. as $P_{t=0} = \rho I_p$, $\hat{\theta}_{t=0} = 0$

Step 2. Update P_t and K_t

$$P_t^{-1} = P_{t-1}^{-1} + \varphi_t \Lambda \varphi_t^T$$

$$K_t = P_t \varphi_t \Lambda$$

Update estimate

$$\hat{\theta}_t = \hat{\theta}_{t-1} + K_t \left(y_t - \varphi_t \hat{\theta}_{t-1} \right)$$

e) An estimate of θ is the mean of the observations $y_k \; \forall \; k=1,2,\ldots,t$

$$\underline{\underline{\hat{\theta}_t = \frac{1}{t}\sum_{k=1}^{t} y_k}} \quad \text{(estimate of } \theta\text{)}$$

Divide the sum

$$\hat{\theta}_t = \frac{1}{t}\left(\sum_{k=1}^{t-1} y_k + y_t\right)$$

This gives

$$\underline{\underline{\hat{\theta}_t = \hat{\theta}_{t-1} + \frac{1}{t}\left(y_t - \hat{\theta}_{t-1}\right)}}$$

Task 2

a) Given 8 impulse response matrices
$H_k = DA^{k-1}B \quad \forall\, k=1,2,\dots,8$, $L=2$

$$H_{2|L} = \begin{bmatrix} H_2 & H_3 & H_4 & H_5 & H_6 & H_7 \\ H_3 & H_4 & H_5 & H_6 & H_7 & H_8 \end{bmatrix} = O_2 A C_6$$

where

$$O_2 = \begin{bmatrix} D \\ DA \end{bmatrix}, \quad C_6 = \begin{bmatrix} B & AB & A^2B & A^3B & A^4B & A^5B \end{bmatrix}$$

$$H_{1|L} = \begin{bmatrix} H_1 & H_2 & H_3 & H_4 & H_5 & H_6 \\ H_2 & H_3 & H_4 & H_5 & H_6 & H_7 \end{bmatrix} = O_2 C_6$$

- System order from the SVD of $H_{1|L}$

$$H_{1|L} = USV^T = [U_1\ U_2] \begin{bmatrix} S_1 & 0 \\ 0 & S_2 \end{bmatrix} [V_1\ V_2]^T$$
$$\approx U_1 S_1 V_1^T \quad \text{when} \quad S_2 \approx 0$$

Comparing gives e.g. the output normal realization

$\underline{\underline{O_L = U_1}}$ and $\underline{\underline{C_J = S_1 V_1^T}}$

where $S_1 \in \mathbb{R}^{n \times n}$, $\underbrace{n = \text{rank}(H_{1|L})}_{\text{singular values}} = \#$ of

- $H_{1|L} = O_L C_y$ and $H_{2|L} = O_L A C_y$

where O_L is the extended observability matrix, and C_y the extended controllability matrix.)

b) - $\bar{y}_k = D\bar{x}_k + Eu_k$ (prediction of y_k)

- $\hat{x}_k = \bar{x}_k + K(y_k - \bar{y}_k)$ (aposteriori estimate)

$\bar{x}_{k+1} = A\hat{x}_k + Bu_k$ (update apriori estimate)

- From $E(\Delta\bar{x}_{k+1}\varepsilon_k) = AE(\Delta\hat{x}_k\varepsilon_k) = 0$ we find

$$\underline{\underline{K = \bar{X} D^T (D\bar{X}D^T + W)^{-1}}}$$

- Inovations form by eliminating $\hat{x}_k$

$$\begin{aligned} \bar{x}_{k+1} &= A\bar{x}_k + Bu_k + \tilde{K}\varepsilon_k \\ y_k &= \bar{y}_k + \varepsilon_k \end{aligned}$$

where $\underline{\underline{\tilde{K} = AK}}$

c) For constant kalman gain

$$\bar{y}_k = g(\bar{x}_k)$$

$$\hat{x}_k = \bar{x}_k + K(y_k - \bar{y}_k)$$

$$\bar{x}_{k+1} = f(\hat{x}_k, u_k)$$

Updating K and covariance matrices

① $\bar{y}_k = g(\bar{x}_k)$

② $D_k = \left. \dfrac{d\, g(x_k, u_k)}{d x_k} \right|_{\bar{x}_k, u_k}$

$$K_k = \bar{X}_k D_k (D_k \bar{X}_k D_k^T + W)^{-1}$$

③ $\hat{x}_k = \bar{x}_k + k_k (y_k - \bar{y}_k)$

④ $\bar{x}_{k+1} = f(\hat{x}_k, u_k)$

⑤ $A_k = \left. \dfrac{d f(x_k, u_k)}{d x_k} \right|_{\hat{x}_k, u_k}$

$$\hat{X}_k = (I - K_k D_k) \bar{X}_k (I - K_k D_k)^T + K_k W K_k^T$$

$$\bar{X}_{k+1} = A_k \hat{X}_k A_k^T + V$$

d) • Kalman fiter on prediction form

$$\bar{x}_{k+1} = A\bar{x}_k + Bu_k + K(y_k - \bar{y}_k)$$

$$\bar{y}_k(\theta) = D\bar{x}_k + Eu_k$$

• Prediction error

$$\varepsilon_k = y_k - \bar{y}_k(\theta)$$

• $A = \begin{bmatrix} 0 & 1 \\ \theta_1 & \theta_2 \end{bmatrix}, B = \begin{bmatrix} \theta_3 \\ \theta_4 \end{bmatrix}, K = \begin{bmatrix} \theta_5 \\ \theta_6 \end{bmatrix}$

$D = \begin{bmatrix} 1 & 0 \end{bmatrix}, E = \theta_7, \bar{x}_{k=0} = \begin{bmatrix} \theta_8 \\ \theta_9 \end{bmatrix}$

$$\theta = [\theta_1\ \theta_2\ \cdots\ \theta_9]^T \in \mathbb{R}^p, \quad p = 9$$

Task 3

a) $N=10$, $L=2$ and $g=0$, $y_k \; \forall k=0,1,\dots,9$

$$Y_{0|3} = O_3 X_0 + H_3^d U_{0|2} \qquad (1)$$

We have

$$Y_{0|3} = \begin{bmatrix} y_0 & y_1 & y_2 & y_3 & y_4 & y_5 & y_6 & y_7 \\ y_1 & y_2 & y_3 & y_4 & y_5 & y_6 & y_7 & y_8 \\ y_2 & y_3 & y_4 & y_5 & y_6 & y_7 & y_8 & y_9 \end{bmatrix} \in \mathbb{R}^{3m \times 8}$$

$K = 8$ columns in $Y_{0|3}$, X_0 and $U_{0|2}$

$$X_0 = \begin{bmatrix} x_0 & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 \end{bmatrix}, \quad O_3 = \begin{bmatrix} D \\ DA \\ DA^2 \end{bmatrix}$$

$$U_{0|2} = \begin{bmatrix} u_0 & u_1 & u_2 & u_3 & u_4 & u_5 & u_6 & u_7 \\ u_1 & u_2 & u_3 & u_4 & u_5 & u_6 & u_7 & u_8 \end{bmatrix}$$

- $g=0$ means that $E=0$

$$H_3^d = \begin{bmatrix} 0 & 0 \\ DB & 0 \\ DAB & DB \end{bmatrix}$$

b) Multiply from right in (1) with projection matrix

$$\underline{\underline{P = U_{0|2}^{\perp} = I_K - U_{0|2}^T \left(U_{0|2} U_{0|2}^T \right)^{+} U_{0|2}}}$$

NB Many students miss this definition

such that $U_{0|2} P = U_{0|2} U_{0|2}^{\perp}$

$$= U \left(I - U^T (U U^T)^{+} U \right) = U - U \underline{\underline{= 0}}$$

Then

$$\underline{\underline{Z_{0|L+1} = Y_{0|L+1} P = Y_{0|L+1} U_{0|L+g}^{\perp} = Y_{0|3} U_{0|2}}}$$

c) • SVD of $Z_{0|L+1}$

$$Z_{0|L+1} = U S V^T = [U_1 \; U_2] \begin{bmatrix} S_1 & 0 \\ 0 & S_2 \end{bmatrix} [V_1 \; V_2]^T \approx U_1 S_1 V_1^T$$

• System order n = # of non-zero singular values in $Z_{0|L+1}$

• $\underline{\underline{O_{L+1} = U_1}}$

• D as the upper $m \times n$ submatrix in O_{L+1}

- A may be found using the shift invariance principle, when we know O_{L+1} and noticing that

$$O_{L+1} = \begin{bmatrix} O_L \\ \hline X \end{bmatrix} = \begin{bmatrix} \overset{D}{X} \\ \hline O_L A \end{bmatrix} \qquad (2)$$

- Define $\underline{O}_{L+1}$ as O_{L+1} omitting the last $m \times n$ submatrix (indicated as X above)

$\Rightarrow \underline{O}_{L+1} = O_L$

- Define $\overline{O}_{L+1}$ as O_{L+1} omitting the upper $(m \times m)$ submatrix in O_{L+1} (se (2) above)

$\Rightarrow \overline{O}_{L+1} = O_L A = \underline{O}_{L+1} A$

- This gives

$$\underline{\underline{A = (\underline{O}_{L+1}^T \underline{O}_{L+1})^{-1} \underline{O}_{L+1}^T \overline{O}_{L+1} = \underline{O}_{L+1}^{+} \overline{O}_{L+1}}}$$

d)

- At this stage $\tilde{A}_L$ is known

$$\tilde{A}_L = O_L A (O_L^T O_L)^{-1} O_L^T$$

- Compute $\hat{B}_L$ from (17)

$$\tilde{B}_L = (Y_{1|L} - \tilde{A}_L Y_{0|L}) U_{0|L+g}^T (U_{0|L+g} U_{0|L+g}^T)^+$$

Now using the structure of $\hat{B}_L$ to find E and $O_L B$

$$\hat{B}_L = [O_L B \ \ H_L^d] - \tilde{A}_L [H_L^d \ \ 0]$$

Take the example to illustrate

- We find

$$\tilde{B}_2 = \left[\begin{bmatrix} D \\ DA \end{bmatrix} B - \tilde{A}_2 \begin{bmatrix} 0 \\ DB \end{bmatrix} \;\vdots\; \begin{bmatrix} 0 \\ DB \end{bmatrix} \right]$$

and $\begin{bmatrix} 0 \\ DB \end{bmatrix}$ directly as the last column in $\hat{B}_L$. Then we solve the 1st column in $\tilde{B}_2$ for $O_2 B = \begin{bmatrix} D \\ DA \end{bmatrix} B$ and then we find B.

Task 4

a) Parameters: $N=10$, $L=J=2$, $g=0$ $(E=0)$

Start with matrices in (22).

$$Y_{J+1|L} = Y_{3|2} = \begin{bmatrix} y_3 & y_4 & y_5 & y_6 & y_7 & y_8 \\ y_4 & y_5 & y_6 & y_7 & y_8 & y_9 \end{bmatrix}$$

$K=6$ columns in all data matrices!

$$Y_{J|L} = \begin{bmatrix} y_2 & y_3 & y_4 & y_5 & y_6 & y_7 \\ y_3 & y_4 & y_5 & y_6 & y_7 & y_8 \end{bmatrix}$$

$$U_{J|L+g} = U_{2|2} = \begin{bmatrix} u_2 & u_3 & u_4 & u_5 & u_6 & u_7 \\ u_3 & u_4 & u_5 & u_6 & u_7 & u_8 \end{bmatrix}$$

$$E_{J|L+1} = E_{2|3} = \begin{bmatrix} e_2 & e_3 & e_4 & e_5 & e_6 & e_7 \\ e_3 & e_4 & e_5 & e_6 & e_7 & e_8 \\ e_4 & e_5 & e_6 & e_7 & e_8 & e_9 \end{bmatrix}$$

$$\tilde{A}_L = O_L A (O_L^T O_L)^{-1} O_L^T, \quad \tilde{B}_L = [O_L B \;\; H_L^d] - \tilde{A}_L [H_L^d \;\; 0]$$

$$\tilde{C}_J = [O_L C \;\; H_L^s] - \tilde{A}_L [H_L^s \;\; 0]$$

$$H_L^d = H_2^d = \begin{bmatrix} 0 \\ DB \end{bmatrix}, \quad H_L^s = \begin{bmatrix} F & 0 \\ DC & F \end{bmatrix}$$

Matrices in (2)

$$Y_{g|L} = Y_{2|2} = \begin{bmatrix} y_2 & y_3 & y_4 & y_5 & y_6 & y_7 \\ y_3 & y_4 & y_5 & y_6 & y_7 & y_8 \end{bmatrix}$$

$$X_g = X_2 = \begin{bmatrix} x_2 & x_3 & x_4 & x_5 & x_6 & x_7 \end{bmatrix}$$

$$O_L = \begin{bmatrix} D \\ DA \end{bmatrix}$$

$$U_{g|L+g-1} = U_{2|1} = \begin{bmatrix} u_2 & u_3 & u_4 & u_5 & u_6 & u_7 \end{bmatrix}$$

$$E_{g|L} = E_{2|2} = \begin{bmatrix} e_2 & e_3 & e_4 & e_5 & e_6 & e_7 \\ e_3 & e_4 & e_5 & e_6 & e_7 & e_8 \end{bmatrix}$$

b) We may put $g=0$ in a deterministic system and when ~~no~~ noise!

$$Y_{0|L} = O_L X_0 + H_L^d U_{0|L+g-1} \qquad (3)$$

$$Y_{1|L} = \hat{A}_L Y_{0|L} + \hat{B}_L U_{0|L+g} \qquad (4)$$

Multiply (3) and (4) with projection matrix, such that

$$U^{\perp}_{0|L+g} = P = I - U_x^T (U_x U_x^T)^+ U_x$$

when x represents indexes $0|L+g$!

Hence,

$$\underline{\underline{Z_{0|L} = Y_{0|L} U^{\perp}_{0|L+g}}} = O_L X_g^a$$

when $X_g^a = X_g U^{\perp}_{0|L+g}$

and

$$\underline{\underline{\underbrace{Y_{1|L} U^{\perp}_{0|L+g}}_{Z_{1|L}}}} = \underline{\underline{\hat{A}_L \underbrace{Y_{0|L} U^{\perp}_{0|L+g}}_{= Z_{0|L}}}}$$

For a general system we first remove the noise term by projecting down on

$$W = \begin{bmatrix} U_{y|k+g} \\ U_{0|y} \\ Y_{0|y} \end{bmatrix}$$

and remove deterministic term with $U_{0|k+g}^{\perp}$

$$\left(Y_{y|k} / W\right) U_{0|k+g}^{\perp} = O_k X_y^{a} = Z_{y|k}$$

$$\underbrace{\left(Y_{y+1|k} / W\right) U_{0|k+g}^{\perp}}_{= Z_{y+1|k}} = \tilde{A}_k \underbrace{\left(Y_{y|k} / W\right) U_{0|k+g}^{\perp}}_{Z_{y|k}}$$

and were the projection A/B is defined as

$$\underline{\underline{A/B = AB^T(BB^T)^+B}}$$

Should be defined

when $(\ )^+$ denotes the pseudo inverse!

c) • n and O_L from svd of

$$Z_{J|L} = USV^T \approx U_1 S_1 V_1^T$$

• n the number of non-zero singular values in $Z_{J|L}$, dimension of $S_1 \in \mathbb{R}^{n \times n}$

• $O_L = U_1$

• A from $Z_{J+1|L} = O_L A O_L^+ Z_{J|L}$

gives $Z_{J+1|L} = U_1 A \underbrace{U_1^+ \cdot U_1}_{I} S_1 V_1^T$

$$Z_{J+1|L} = U_1 A S_1 V_1^T \Rightarrow Z_{J+1|L} V_1 = U_1 A S_1$$

$$\Rightarrow \underline{\underline{A = U_1^T Z_{J+1|L} V_1 S_1^{-1}}}$$

d) Comparing the two variants of the Kalman filter we obtain

$$C e_k = K \varepsilon_k$$

$$F e_k = \varepsilon_k$$

and $E(e_k e_k^T) = I$.

Gives

$$e_k = F^{-1} \varepsilon_k$$

and

$$C e_k = C F^{-1} \varepsilon_k = K \varepsilon_k$$

and hence

$$\underline{\underline{K = C F^{-1}}}$$

when the Kalman filter exist!

e). • We may estimate the innovations process $\varepsilon_k = Fe_k$ for $k \geq J$ with the projection

$$Z^{S}_{J|1} = Y_{J|1} - Y_{J|1} / \begin{bmatrix} U_{0|J} \\ Y_{0|J} \end{bmatrix}$$

$$= [\varepsilon_J \; \varepsilon_{J+1} \; \cdots \; \varepsilon_{N-1}]$$

• And when ε_k is known we simply may solve a deterministic subspace problem as in Task 3

$$x_{k+1} = A x_k + \tilde{B} \tilde{u}_k$$
$$\tilde{y}_k = D x_k \cancel{+}$$

where $\tilde{y}_k = y_k - \varepsilon_k$, $\tilde{u}_k = \begin{bmatrix} u_k \\ \varepsilon_k \end{bmatrix}$

$\tilde{B} = [B \; K]$, and notice that $E = 0$ when feedback in the data!