Final Exam SCE2202
System identification and optimal estimation
Monday June 3, 2010 Time: kl. 9.00 - 15.00

The final exam consists of: 4 tasks.
The exam counts 70% of the final grade.
Available aids: pen and paper

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Task 1 (16%): Deterministic Subspace System Identification

Consider the discrete time deterministic model, ie.

$$x_{k+1} = Ax_k + Bu_k, \tag{1}$$
$$y_k = Dx_k + Eu_k, \tag{2}$$

where the following output and input data matrices are known

$$Y = \begin{bmatrix} y_0^T \\ y_1^T \\ \vdots \\ y_{N-1} \end{bmatrix} \in \mathbb{R}^{N\times m}, \; U = \begin{bmatrix} u_0^T \\ u_1^T \\ \vdots \\ u_{N-1} \end{bmatrix} \in \mathbb{R}^{N\times r}. \tag{3}$$

a) Based on the model in Equations (1) and (2) and with known data as given in (3) we can develop the following matrix equations

$$Y_{0|L} = O_L X_0 + H_L^d U_{0|L+g-1}, \tag{4}$$
$$Y_{1|L} = \tilde{A}_L Y_{0|L} + \tilde{B}_L U_{0|L+g}, \tag{5}$$

where $L \geq 1$ are user specified positive integer numbers.

Write down the structure of the matrices in the matrix equations, (4) and (5), with parameters $N = 9$, $L = 2$, $J = 2$ and $g = 0$.

b) By using (3) and Equations (4) and (5) we may formulate the equations

$$Z_{0|L} = O_L X_0^a \tag{6}$$

and

$$Z_{1|L} = \tilde{A}_L Z_{0|L} \tag{7}$$

Find expressions for the data matrices $Z_{0|L}$ and $Z_{1|L}$.

Remark: define the projections which is involved in the expressions for $Z_{1|L}$ and $Z_{0|L}$.

c) Show how

- the system order, n
- the extended observability matrix O_L
- the system matrices A and D

can be estimated.

d) Assume that the system is single output. Is it then possible to write the deterministic system as a linear regression model? , i.e., as a model

$$y_k = \varphi_k^T \theta \tag{8}$$

where φ_k contains the regressors and θ is a vector of parameters. The answer is YES? or NO?

Task 2 (16%): Subspace System Identification of combined deterministic and stochastic systems

Consider the discrete time model on innovations form, ie.

$$\begin{aligned} x_{k+1} &= Ax_k + Bu_k + Ce_k, && (9) \\ y_k &= Dx_k + Eu_k + Fe_k, && (10) \end{aligned}$$

where e_k is white noise with unit covariance matrix, i.e., $\mathrm{E}(e_k e_k^T) = I$ and where the following output and input data matrices are known

$$Y = \begin{bmatrix} y_0^T \\ y_1^T \\ \vdots \\ y_{N-1} \end{bmatrix} \in \mathbb{R}^{N \times m}, \; U = \begin{bmatrix} u_0^T \\ u_1^T \\ \vdots \\ u_{N-1} \end{bmatrix} \in \mathbb{R}^{N \times r}. \tag{11}$$

a) Based on the model in Equations (9) and (10) and with known data as given in (11) we can develop the following matrix equation

$$Y_{J|L+1} = O_{L+1} X_J + H_{L+1}^d U_{J|L+g} + H_{L+1}^s E_{J|L+1}, \tag{12}$$

where $J \geq 1$ and $L \geq 1$ are user specified positive integer numbers.

By using (11) and the matrix Equation (12)we may formulate the projection equation

$$Z_{J|L+1} = O_{L+1} X_J^a. \tag{13}$$

Find an expression for the data matrix $Z_{J|L+1}$ for the following case:

- a general combined deterministic and stochastic system.

Remark: define the projections which is involved in the expression for $Z_{J|L+1}$.

b) Explain how:

1. the extended observability matrix, O_{L+1}, may be estimated.
2. explain how the "shift invariance principle" may be used to estimate the A matrix, i.e., explain how A may be estimated from O_{L+1}.

c) Consider now the following Kalman filter on innovations form

$$\begin{aligned} x_{k+1} &= Ax_k + Bu_k + K\varepsilon_k, \quad (14) \\ y_k &= Dx_k + Eu_k + \varepsilon_k \quad (15) \end{aligned}$$

What is the relationship between the Kalman filter on innovations form in Equations (14) and (15) and the innovations formulation in Equations (9) and (10)?

d) Put $L = 0$ and $g = 0$ in Eq. (12) and use that

$$X_J = X_J / \begin{bmatrix} Y_{0|J} \\ U_{0|J} \end{bmatrix}, \quad (16)$$

for "large" J.

- Show how the innovations sequence $\varepsilon_k = Fe_k \ \forall \ k = J, \cdots, N-1$ may be estimated.
- How can you use the estimated innovations sequence, ε_k, for system identification of closed loop systems?

Task 3 (10%): Prediction error methods

A Kalman filter on innovations form for a linear discrete time system is given by

$$\begin{aligned} \bar{x}_{k+1} &= A\bar{x}_k + Bu_k + Ke_k, && (17) \\ y_k &= D\bar{x}_k + e_k && (18) \end{aligned}$$

where $\bar{x}_k$ is the predicted state, $\bar{x}_1$ is the initial state, $y_k \in \mathbb{R}^m$ is the measurement vector and e_k is the innovations process.
We will assume that the following input and output data are known

$$\left.\begin{matrix} u_k \\ y_k \end{matrix}\right\} \; \forall \; k = 1, \ldots, N \qquad (19)$$

a) Write down a Kalman filter on prediction form, i.e. the filter used to compute the predicted measurement, $\bar{y}_k$, of the measurement y_k.

b)

- What is a parameter vector, θ?
- Give an example of the relationship between the parameter vector θ and the prediction formulation of a Kalman filter for a single output and single input dynamic system in state space model form, as in Eqs. (17) and (18) with $n = 3$ states.
- How many parameters, p, is it in this parameter vector, $\theta \in \mathbb{R}^p$?

Task 4 (16%): Ordinary Least Squares method and recursive system identification

a) Given a system

$$\begin{bmatrix} x^1_{k+1} \\ x^2_{k+1} \end{bmatrix} = \begin{bmatrix} a_1 & 1 \\ a_2 & 0 \end{bmatrix} \begin{bmatrix} x^1_k \\ x^2_k \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} u_k + \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} e_k \tag{20}$$

$$y_k = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x^1_k \\ x^2_k \end{bmatrix} + e_k \tag{21}$$

For which values of the parameters k_1 and k_2 may the above model be written as an ARX model and hence as a linear regression model of the form

$$y_k = \varphi_k^T \theta_0 + e_k \tag{22}$$

- In particular, define the regression vector, φ_k, and the parameter vector, θ_0.
- Also define the parameters k_1 and k_2.

b)

- Based on the regression model in step a) above, find a predictor, $\bar{y}_k(\theta)$, for the measurement y_k.
- Define the prediction error, ε_k.

c) Consider the following prediction error criterion

$$V_N(\theta) = \frac{1}{N} \sum_{k=1}^{N} \varepsilon_k^T \Lambda \varepsilon_k \tag{23}$$

where Λ is a specified and symmetric weighting matrix.

Find the Ordinary Least Squares (OLS) estimate, $\hat{\theta}_N$, of the true parameter vector θ_0.

d) What is a so called BLUE estimator?

e) Based on the OLS solution in step c) above, show how we can develop a recursive Ordinary Least Squares (ROLS) method of the following form

$$\hat{\theta}_t = \hat{\theta}_{t-1} + K_t(y_t - \varphi_t^T \hat{\theta}_{t-1}) \tag{24}$$

You shall in particular find equations for computing the gain, K_t, in Equation (24).

Task 5 (12%): Realization theory and the Kalman filter

a) Assume known impulse responses

$$H_k = DA^{k-1}B \;\forall\; k = 1, \cdots, 7. \tag{25}$$

Answer the following:

- Write up the Hankel matrices $\mathbf{H}_{1|L}$ and $\mathbf{H}_{2|L}$ where you should use $L = 2$.
- Show how you can find the system order, n, the extended observability matrix O_L and the extended controllability matrix C_J from a Singular Value decomposition (SVD) of $\mathbf{H}_{1|L}$.
- How are $\mathbf{H}_{1|L}$ and $\mathbf{H}_{2|L}$ related to O_L and C_J ?

b) Given a system modelled by a discrete time, state space model as follows

$$x_{k+1} = Ax_k + Bu_k + v_k, \tag{26}$$
$$y_k = Dx_k + Eu_k + w_k, \tag{27}$$

where v_k is white process noise and w_k is white measurements noise. Assume that the noise are uncorrelated, i.e. $\mathrm{E}(v_k w_k^T) = 0$.

- Write down a Kalman filter on apriori-aposteriori form for optimal estimation of the state vector x_k.
- Find a formula for the Kalman filter gain matrix, K, in the Kalman filter on apriori-aposteriori form.
- Show how the apriori-aposteriori formulation of the Kalman filter kan be written as a Kalman filter on innovations form.

c) Given a non-linear system

$$x_{k+1} = f(x_k, u_k) + v_k \tag{28}$$
$$y_k = g(x_k) + w_k \tag{29}$$

where v_k and w_k are discrete white process noise and discrete white measurements noise, respectively.

Formulate the Kalman filter on apriori-aposteriori form for the non-linear system model in Equations (28) and (29).

d) Why is a Kalman-filter often called an optimal estimator?