Final Exam SCE2202
System identification and optimal estimation
Monday 4. July 2008 Time: kl. 9.00 - 12.00

The final exam consists of: 4 tasks.
The exam counts 70% of the final grade.
Available aids: pen and paper

Teacher: PhD David Di Ruscio
Systems and Control Engineering
Department of technology
Telemark University College
N-3914 Porsgrunn

Task 1 (20%): Subspace System Identification

Consider the discrete time model on innovations form, ie.

$$
\begin{aligned}
x_{k+1} &= Ax_k + Bu_k + Ce_k, && (1)\\
y_k &= Dx_k + Eu_k + Fe_k && (2)
\end{aligned}
$$

where e_k is white noise with unit covariance matrix, i.e., $\mathrm{E}(e_k e_k^T) = I$ and where the following output and input data matrices are known

$$
Y = \begin{bmatrix} y_0^T \\ y_1^T \\ \vdots \\ y_{N-1} \end{bmatrix} \in \mathbb{R}^{N\times m}, \; U = \begin{bmatrix} u_0^T \\ u_1^T \\ \vdots \\ u_{N-1} \end{bmatrix} \in \mathbb{R}^{N\times r}. \tag{3}
$$

a) Based on the model in Equations (1) and (2) and with known data as given in (3) we can develop the following matrix equations

$$
\begin{aligned}
Y_{J|L} &= O_L X_J + H_L^d U_{J|L+g-1} + H_L^s E_{J|L}, && (4)\\
Y_{J+1|L} &= \tilde{A}_L Y_{J|L} + \tilde{B}_L U_{J|L+g} + \tilde{C}_L E_{J|L+1}, && (5)
\end{aligned}
$$

where $J \geq 1$ and $L \geq 1$ are user specified positive integer numbers.

Write down the structure of the matrices in the matrix equations, (4) and (5), with parameters $N = 9$, $L = 2$, $J = 2$ and $g = 0$.

b) By using (3) and Equations (4) and (5) we may formulate the equations

$$
Z_{J|L} = O_L X_J^a \tag{6}
$$

and

$$
Z_{J+1|L} = \tilde{A}_L Z_{J|L} \tag{7}
$$

Find expressions for the data matrices $Z_{J|L}$ and $Z_{J+1|L}$ for the following two cases:

- a deterministic system, i.e., when $e_k = 0$.
- a general combined deterministic and stochastic system.

Remark: define the projections which is involved in the expressions for $Z_{J+1|L}$ and $Z_{J|L}$.

c) Show how

- the system order, n
- the extended observability matrix O_L
- the system matrices A and D

can be estimated.

d) Consider now the following Kalman filter on innovations form

$$\begin{aligned} x_{k+1} &= Ax_k + Bu_k + K\varepsilon_k, && (8) \\ y_k &= Dx_k + Eu_k + \varepsilon_k && (9) \end{aligned}$$

What is the relationship between the Kalman filter on innovations form in Equations (8) and (9) and the innovations formulation in Equations (1) and (2)?

e) Consider that the known input and output data as given in (3) are collected in closed loop, i.e., we assume that there is feedback in the known data. Consider the projection

$$\begin{aligned} Z^s_{J|1} &= FE_{J|1} = Y_{J|1} - Y_{J|1} / \begin{bmatrix} U_{0|J} \\ Y_{0|J} \end{bmatrix} \\ &= \begin{bmatrix} Fe_J & Fe_{J+1} & \dots & Fe_{N-1} \end{bmatrix} \\ &= \begin{bmatrix} \varepsilon_J & \varepsilon_{J+1} & \dots & \varepsilon_{N-1} \end{bmatrix} && (10) \end{aligned}$$

Explain how this projection can be used in order to develop a subspace identification algorithm for closed loop systems.

Task 2 (20%): Prediction error methods

A Kalman filter on innovations form for a linear discrete time system is given by

$$\bar{x}_{k+1} = A\bar{x}_k + Bu_k + Ke_k, \tag{11}$$

$$y_k = D\bar{x}_k + e_k \tag{12}$$

where $\bar{x}_k$ is the predicted state, $\bar{x}_1$ is the initial state, $y_k \in \mathbb{R}^m$ is the measurement vector and e_k is the innovations process.
We will assume that the following input and output data are known

$$\left.\begin{array}{c} u_k \\ y_k \end{array}\right\} \forall\ k = 1, \ldots, N \tag{13}$$

a) Write down a Kalman filter on prediction form, i.e. the filter used to compute the predicted measurement, $\bar{y}_k$, of the measurement y_k.

b)

- What is a parameter vector, θ?
- Give an example of the relationship between the parameter vector θ and the prediction formulation of a Kalman filter for a single output and single input dynamic system in state space as in (11) and (12) with $n = 2$ states.
- How many parameters, p, is it in this parameter vector, $\theta \in \mathbb{R}^p$?

c) Define the prediction error, ε_k, as a function of y_k and $\bar{y}_k$.

d) Define and answer the following questions:

- Give an example of a prediction error criterion for both a single output system ($m = 1$) and a multiple output system ($m > 1$.
- Describe formally how the optimal parameter estimate, $\hat{\theta}_N$, can be computed.

e) Try to formulate a formula (an iteration scheme) which may be used to find the optimal parameter estimate. This formula may be deduced from the Newton-Raphson method.

Task 3 (15%): Ordinary Least Squares method and recursive system identification

a) Given a system

$$\begin{bmatrix} x^1_{k+1} \\ x^2_{k+1} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ a_1 & a_2 \end{bmatrix} \begin{bmatrix} x^1_k \\ x^2_k \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} u_k + \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} e_k \tag{14}$$

$$y_k = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x^1_k \\ x^2_k \end{bmatrix} + e_k \tag{15}$$

For which values of the parameters k_1 and k_2 may the above model be written as an ARX model and hence as a linear regression model of the form

$$y_k = \varphi_k^T \theta_0 + e_k \tag{16}$$

- In particular, define the regression vector, φ_k, and the parameter vector, θ_0.
- Also define the parameters k_1 and k_2.

b)

- Based on the regression model in step a) above, find a predictor, $\bar{y}_k(\theta)$, for the measurement y_k.
- Define the prediction error, ε_k.

c) Consider the following prediction error criterion

$$V_N(\theta) = \frac{1}{N} \sum_{k=1}^{N} \varepsilon_k^T \Lambda \varepsilon_k \tag{17}$$

where Λ is a specified and symmetric weighting matrix.

Find the Ordinary Least Squares (OLS) estimate, $\hat{\theta}_N$, of the true parameter vector θ_0.

d) What is a so called BLUE estimator?

e) Based on the OLS solution in step c) above, show how we can develop a recursive Ordinary Least Squares (ROLS) method of the following form

$$\hat{\theta}_t = \hat{\theta}_{t-1} + K_t (y_t - \varphi_t^T \hat{\theta}_{t-1}) \tag{18}$$

You shall in particular find equations for computing the gain, K_t, in Equation (18).

Task 4 (15%): Realization theory and the Kalman filter

a) Assume known impulse responses

$$H_k = DA^{k-1}B \ \forall \ k = 1, \cdots, 7. \tag{19}$$

Answer the following:

- Write up the Hankel matrices $\mathbf{H}_{1|L}$ and $\mathbf{H}_{2|L}$ where you should use $L = 2$.
- Show how you can find the system order, n, the extended observability matrix O_L and the extended controllability matrix C_J from a Singular Value decomposition (SVD) of $\mathbf{H}_{1|L}$.
- How are $\mathbf{H}_{1|L}$ and $\mathbf{H}_{2|L}$ related to O_L and C_J ?

b) Given a system modelled by a discrete time, state space model as follows

$$x_{k+1} = Ax_k + Bu_k + v_k, \tag{20}$$

$$y_k = Dx_k + Eu_k + w_k, \tag{21}$$

where v_k is white process noise and w_k is white measurements noise. Assume that the noise are uncorrelated, i.e. $\mathrm{E}(v_k w_k^T) = 0$.

- Write down a Kalman filter on apriori-aposteriori form for optimal estimation of the state vector x_k.
- Find a formula for the Kalman filter gain matrix, K, in the Kalman filter on apriori-aposteriori form.
- Show how the apriori-aposteriori formulation of the Kalman filter kan be written as a Kalman filter on innovations form.

c) Given a non-linear system

$$x_{k+1} = f(x_k, u_k) + v_k \tag{22}$$

$$y_k = g(x_k) + w_k \tag{23}$$

where v_k and w_k are discrete white process noise and discrete white measurements noise, respectively.

Formulate the Kalman filter on apriori-aposteriori form for the non-linear system model in Equations (22) and (23).

d) Why is a Kalman-filter often called an optimal estimator?