Final Exam SCE2202
System identification and optimal estimation
Monday 4. July 2008 Time: kl. 9.00 - 12.00

The final exam consists of: 4 tasks.
The exam counts 70% of the final grade.
Available aids: pen and paper

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Task 1 (20%): Prediction error methods

A Kalman filter on innovations form for a linear discrete time system is given by

$$\bar{x}_{k+1} = A\bar{x}_k + Bu_k + Ke_k, \tag{1}$$
$$y_k = D\bar{x}_k + e_k \tag{2}$$

where $\bar{x}_k$ is the predicted state, $\bar{x}_1$ is the initial state, $y_k \in \mathbb{R}^m$ is the measurement vector and e_k is the innovations process.
We will assume that the following input and output data are known

$$\left.\begin{array}{c} u_k \\ y_k \end{array}\right\} \;\forall\; k = 1, \ldots, N \tag{3}$$

a) Write down a Kalman filter on prediction form, i.e. the filter used to compute the predicted measurement, $\bar{y}_k$, of the measurement y_k.

b)

- What is the meaning of a parameter vector, θ?
- Give an example of the relationship between the parameter vector θ and the prediction formulation of a Kalman filter for a single output dynamic system in state space as in (1) and (2) with $n = 3$ states.
- How many parameters, p, is it in this parameter vector, $\theta \in \mathbb{R}^p$?

c) Define the prediction error, ε_k, as a function of y_k and $\bar{y}_k$.

d) Define and answer the following questions:

- What is a prediction error criterion, $V_N(\theta)$.
- Give an example of a prediction error criterion for both a single output system ($m = 1$) and a multiple output system ($m > 1$.
- Describe how the optimal parameter estimate, $\hat{\theta}_N$, can be computed.

Task 2 (20%): Ordinary Least Squares method and recursive system identification

a) Given a system

$$\begin{bmatrix} x_{k+1}^1 \\ x_{k+1}^2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ a_1 & a_2 \end{bmatrix} \begin{bmatrix} x_k^1 \\ x_k^2 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} u_k + \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} e_k \tag{4}$$

$$y_k = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_k^1 \\ x_k^2 \end{bmatrix} + e_k \tag{5}$$

For which values of the parameters c_1 and c_2 may the above model be written as an ARX model and hence as a linear regression model of the form

$$y_k = \varphi_k^T \theta_0 + e_k \tag{6}$$

- In particular, define the regression vector, φ_k, and the parameter vector, θ_0.
- Also define the parameters c_1 and c_2.

b)

- Based on the regression model in step a) above, find a predictor, $\bar{y}_k(\theta)$, for the measurement y_k.
- Define the prediction error, ε_k.

c) Consider the following prediction error criterion

$$V_N(\theta) = \frac{1}{N} \sum_{k=1}^{N} \varepsilon_k^T \Lambda \varepsilon_k \tag{7}$$

where Λ is a specified and symmetric weighting matrix.

Find the Ordinary Least Squares (OLS) estimate, $\hat{\theta}_N$, of the true parameter vector θ_0.

d) Based on the OLS solution in step c) above, show how we can develop a recursive Ordinary Least Squares (ROLS) method of the following form

$$\hat{\theta}_t = \hat{\theta}_{t-1} + K_t (y_t - \varphi_t^T \hat{\theta}_{t-1}) \tag{8}$$

You shall in particular find equations for computing the gain, K_t, in Equation (8).

Task 3 (15%): Model structures

a) Given a polynomial model

$$A(q)y_k = B(q)u_k + C(q)e_k \tag{9}$$

where $A(q)$, $B(q)$ and $C(q)$ are polynomials in q^{-1}.

What is the common name for this model structure?

b) Write down the structure of an ARX model, in the same form as in Equation (9).

c) Write down the structure of an OE model, in the same form as in Equation (9).

d) Write the following state space model, i.e.,

$$x_{k+1} = ax_k + bu_k + ke_k, \tag{10}$$

$$y_k = x_k + e_k. \tag{11}$$

on polynomial form as in Equation (9)and specify the necessary polynomials $A(q)$, $B(q)$, ... etc.

- What is the common name of the resulting polynomial model?
- For which value of the parameter k will this model become an ARX model?
- For which value of the parameter k will this model become an OE model?

Note that q^{-1} is the backward shift operator such that $q^{-1}y_k = y_{k-1}$.

Task 4 (15%): Various questions

a) Assume given the ARMAX model

$$y_k = ay_{k-1} + bu_{k-1} + e_k + (k-a)e_{k-1} \tag{12}$$

which is of 1st order. It is known that a general ARMAX model may be approximated as an higher order ARX model.

1. Find a 2nd order ARX approximation to (12). Tips: put $k := k+1$ in (12) and substitute for e_k solved from (12) in this equation gives a 2nd order polynomial model, and assuming $(k-a)^2 \approx 0$ in this equation.
2. Find a 3rd order ARX approximation to (12). Tips: Repetition of step 1 above gives a 3rd order polynomial model, and assume $(k-a)^3 \approx 0$ in this equation.
3. How can the parameters in an ARX model be estimated?

Remarks: The higher order ARX model approximation to a general linear dynamic system may be used in subspace system identification of closed and open loop systems.

b) You should in this sub task find a state space model realization, i.e., obtain the state space model matrices (A, B, D), from the known impulse responses in (**??**), by using Hankel matrix realization theory. I.e. answer the following:

- Write up the Hankel matrices $\mathbf{H}_{1|L}$ and $\mathbf{H}_{2|L}$ where you should use $L = 2$.
- Show how you can find the system order, n, the extended observability matrix O_L and the extended controllability matrix C_L from a Singular Value decomposition (SVD) of $\mathbf{H}_{1|L}$.
- Show how the D and B matrices can be found.
- Show how the A matrix can be found.

c) Given a system modelled by a discrete time, state space model as follows

$$x_{k+1} = Ax_k + Bu_k + v_k, \tag{13}$$

$$y_k = Dx_k + Eu_k + w_k, \tag{14}$$

where v_k is white process noise and w_k is white measurements noise.

- Write down a Kalman filter on apriori-aposteriori form for optimal estimation of the state vector x_k.
- Show how the apriori-aposteriori formulation of the Kalman filter kan be written as a Kalman filter on innovations form.