Master study
Systems and Control Engineering
Department of Technology
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SCE1106 Control Theory

Solution to exercise 2

Solution to task 1

a) A mass balance (conservation of mass) over the tanks gives

$$\frac{d}{dt}(A_1 x_1 \rho) = \rho u_1 - \rho q, \tag{1}$$

$$\frac{d}{dt}(A_2 x_2 \rho) = \rho q + \rho u_2 - \rho v, \tag{2}$$

$$q = k(x_1 - x_2). \tag{3}$$

Since the density is constant it can be cancelled from the equations and we can obtain the equations

$$\dot{x}_1 = -\frac{k}{A_1}x_1 + \frac{k}{A_1}x_2 + \frac{1}{A_1}u_1, \tag{4}$$

$$\dot{x}_2 = \frac{k}{A_2}x_1 - \frac{k}{A_2}x_2 + \frac{1}{A_2}u_2 - \frac{1}{A_2}v. \tag{5}$$

This can be written in matrix form as follows

$$\overbrace{\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix}}^{\dot{x}} = \overbrace{\begin{bmatrix} -\frac{k}{A_1} & \frac{k}{A_1} \\ \frac{k}{A_2} & -\frac{k}{A_2} \end{bmatrix}}^{A} \overbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}^{x} + \overbrace{\begin{bmatrix} \frac{1}{A_1} & 0 \\ 0 & \frac{1}{A_2} \end{bmatrix}}^{B} \overbrace{\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}}^{u} + \overbrace{\begin{bmatrix} 0 \\ -\frac{1}{A_2} \end{bmatrix}}^{C} v \tag{6}$$

b) Putting into numerical values gives the system matrix

$$A = \begin{bmatrix} -0.5 & 0.5 \\ 1 & -1 \end{bmatrix} \tag{7}$$

The eigenvalues (ore poles) for the system matrix are given by

$$\det(sI - A) = 0. \tag{8}$$

This gives the two eigenvalues/poles

$$s_1 = 0 \quad \text{og} \quad s_2 = -\frac{3}{2}. \tag{9}$$

Hence, the system has one time constant

$$T = -\frac{1}{s_2} = \frac{2}{3} \tag{10}$$

and an eigenvalue equal to zero (an eigenvalue in origo) in the complex plane. The pole $s_1 = 0$ represents an integrator in the system. Modeling levels etc. gives typically integrating processes.

c) **Eigenvector for for $\lambda_1 = 0$**
Solving

$$Am_1 = \lambda_1 m_1, \tag{11}$$

where $\lambda_1 = 0$ and

$$m_1 = \begin{bmatrix} m_{11} \\ m_{21} \end{bmatrix}. \tag{12}$$

This gives

$$m_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \tag{13}$$

Eigenvector for $\lambda_1 = -\frac{3}{2}$
Solving

$$Am_2 = \lambda_2 m_2, \tag{14}$$

where $\lambda_1 = -\frac{3}{2}$ and

$$m_2 = \begin{bmatrix} m_{12} \\ m_{22} \end{bmatrix}. \tag{15}$$

This gives

$$m_2 = \begin{bmatrix} 1 \\ -\frac{1}{2} \end{bmatrix}. \tag{16}$$

Hence, an eigenvector for the system is given by

$$M = \begin{bmatrix} m_1 & m_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -\frac{1}{2} \end{bmatrix} \tag{17}$$

e) The transition matrix is then given by

$$\Phi = e^{At} = Me^{\Lambda t}M^{-1} = \begin{bmatrix} \frac{2}{3} + \frac{1}{3}e^{-1.5t} & \frac{1}{3} - \frac{1}{3}e^{-1.5t} \\ \frac{2}{3} - \frac{1}{3}e^{-1.5t} & \frac{1}{3} + \frac{2}{3}e^{-1.5t} \end{bmatrix} \tag{18}$$

f) When $u_1 = u_2 = v = 0$ then the system is described by the autonomous response ore solution given by

$$x(t) = e^{At}x(0), \tag{19}$$

with initial state vector

$$x_0 = x(t=0) = \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \tag{20}$$

Then we have that

$$x(t) = \begin{bmatrix} \frac{4}{3} - \frac{1}{3}e^{-1.5t} \\ \frac{4}{3} + \frac{2}{3}e^{-1.5t} \end{bmatrix}. \tag{21}$$

This means that $x_1(t) = \frac{4}{3} - \frac{1}{3}e^{-1.5t}$ and $x_2(t) = \frac{4}{3} + \frac{2}{3}e^{-1.5t}$.

As we see, both levels will be equal to $\frac{4}{3}$ at steady state, that is when time reach infinity, i.e., when $t \to \infty$. This is also natural from our knowledge of the process physics. The response is plotted in Figure 1.

Figure 1: Time response of the autonomous system $\dot{x} = Ax$ where $x_1(0) = 1$ and $x_2(0) = 2$. This figure is generated by the MATLAB script main_losn2.m

g) The disturbance v is modelled by $v = kx_2$. This can be written in matrix form as

$$v = Gx \tag{22}$$

where

$$G = \begin{bmatrix} 0 & k \end{bmatrix}. \tag{23}$$

Putting this into the state space model gives the autonomous state space model

$$\dot{x} = (A + CG)x \tag{24}$$

where the initial values of the levels are given by

$$x_0 = x(t = 0) = \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}. \tag{25}$$

Hence, the solution is given by

$$x(t) = e^{(A+CG)t} x_0. \tag{26}$$

See the MATLAB script **main_losn2.m** for the simulation of the time response for the state vector $x(t)$. The response is plotted in Figure 2. Note that in this case can not use the transition matrix $\Phi = e^{At}$ which was computed earlier in this exercise.

Figure 2: Time response of the autonomous system $\dot{x} = (A + CG)x$ where $x_1(0) = 1$ and $x_2(0) = 2$. This figure is generated by the MATLAB script main_losn2.m